Addition of Velocities; Four-Velocity

Consider the Lorentz transformation of a coordinate 4-vector:

$$\begin{cases} ct' = \gamma(ct - \vec{\beta}\cdot\vec{r}) \\ \vec{r}_{\parallel}' = \gamma(\vec{r}_{\parallel} - \vec{\beta}ct) \\ \vec{r}_{\perp}' = \vec{r}_{\perp} \end{cases}$$

The corresponding differential expressions are

$$\begin{cases} c\,dt' = \gamma(c\,dt - \vec{\beta}\cdot d\vec{r}) \\ d\vec{r}_{\parallel}' = \gamma(d\vec{r}_{\parallel} - \vec{\beta}c\,dt) \\ d\vec{r}_{\perp}' = d\vec{r}_{\perp} \end{cases}$$

We define
$$\left\{ \begin{array}{ll} \vec{u}_{\parallel} \equiv \dfrac{d\vec{r}_{\parallel}}{dt} & \vec{u}_{\perp} \equiv \dfrac{d\vec{r}_{\perp}}{dt} \\ \vec{u}_{\parallel}' \equiv \dfrac{d\vec{r}_{\parallel}'}{dt'} & \vec{u}_{\perp}' \equiv \dfrac{d\vec{r}_{\perp}'}{dt'} \end{array} \right\}$$

Now $\quad dt' = \gamma(dt - \frac{1}{c}\vec{\beta}\cdot d\vec{r})$.

$$\Rightarrow \quad \frac{d\vec{r}_{\parallel}'}{dt'} = \frac{\gamma(d\vec{r}_{\parallel} - \vec{\beta} c\, dt)}{\gamma(dt - \frac{1}{c}\vec{\beta}\cdot d\vec{r})}$$

or,

$$\boxed{\vec{u}_{\parallel}' = \frac{\vec{u}_{\parallel} - \vec{\beta} c}{1 - \frac{\vec{\beta}\cdot\vec{u}}{c}}}$$

Similarly,

$$\frac{d\vec{r}_{\perp}'}{dt'} = \frac{d\vec{r}_{\perp}}{\gamma(dt - \frac{1}{c}\vec{\beta}\cdot d\vec{r})}$$

or,

$$\boxed{\vec{u}_{\perp}' = \frac{\vec{u}_{\perp}}{\gamma\left(1 - \frac{\vec{\beta}\cdot\vec{u}}{c}\right)}}$$

These transformation laws make it obvious that $d\vec{r}/dt$ is not a component of a 4-vector. However, consider the quantity $d\vec{r}/d\tau$, where $d\tau = dt/\gamma_u$, with τ the invariant proper time.

~~We have $d\tau = dt - \frac{1}{c}\vec{\beta}\cdot d\vec{r}$.~~

$$\left.\begin{aligned} \text{Let} \quad \vec{U} &\equiv \frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt}\frac{dt}{d\tau} \equiv \gamma_u \vec{u} \\ \text{and} \quad U_0 &\equiv \frac{dx_0}{d\tau} = c\frac{dt}{d\tau} \equiv \gamma_u c \end{aligned}\right\} \begin{array}{l}\vec{U} \text{ is} \\ \text{called} \\ \text{proper} \\ \text{velocity}\end{array}$$

Clearly, $\frac{c\,dt'}{d\tau} = \gamma\left(c\frac{dt}{d\tau} - \vec{\beta}\cdot\frac{d\vec{r}}{d\tau}\right)$

or, $$\boxed{U_0' = \gamma(U_0 - \vec{\beta}\cdot\vec{U})}$$

Also, $\frac{d\vec{r}_{\parallel}'}{d\tau} = \gamma\left(\frac{d\vec{r}_{\parallel}}{d\tau} - \vec{\beta}c\frac{dt}{d\tau}\right)$

or, $$\boxed{\vec{U}_{\parallel}' = \gamma(\vec{U}_{\parallel} - \vec{\beta}U_0)}$$

And, $\frac{d\vec{r}_{\perp}'}{d\tau} = \frac{d\vec{r}_{\perp}}{d\tau}$

or, $$\boxed{\vec{U}_{\perp}' = \vec{U}_{\perp}.}$$

Thus, $(U_0, \vec{U}) = (\gamma_u c, \gamma_u \vec{u})$ transforms as a <u>velocity 4-vector</u>.

The inverse transformation for $u_{\parallel}' = \dfrac{u_{\parallel} - \beta c}{1 - \frac{\vec{\beta}\cdot\vec{u}}{c}}$ is

$$u_{\parallel} = \frac{u_{\parallel}' + \beta c}{1 + \frac{\vec{\beta}\cdot\vec{u}'}{c}}.$$

If $\vec{\beta}$ and $\vec{u}'$ are <u>parallel</u>, with $\beta \equiv v/c$, we obtain

$$\boxed{u = \frac{u' + v}{1 + \frac{vu'}{c^2}}},$$

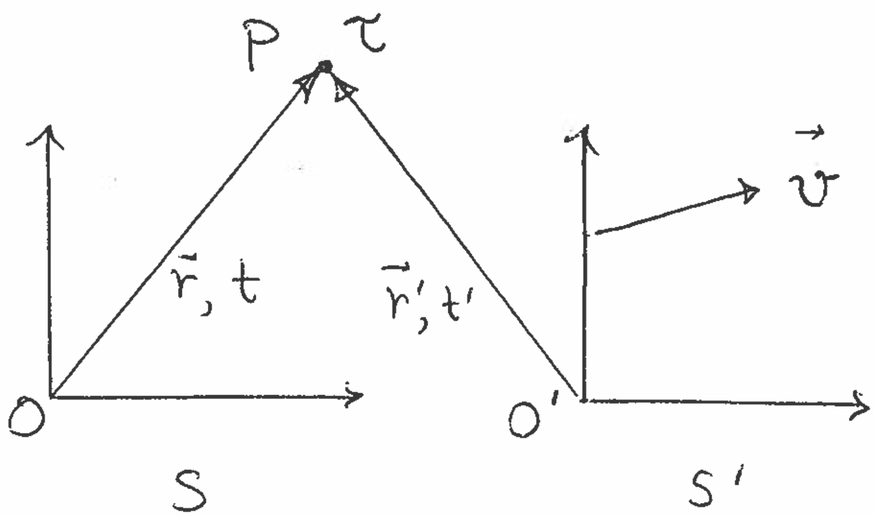
which is the relativistic velocity-addition formula.

If $u' \ll c$ and $v \ll c$, this reduces to the Galilean result, $u \simeq u' + v$.

Suppose $v = v_{max} = c$. Then

$$u = \frac{u' + v}{1 + \frac{v u'}{c^2}} = \frac{u' + c}{1 + \frac{u'}{c}} = c .$$

It is impossible to obtain a speed greater than $\underline{c}$ by adding two velocities. Clearly, if $u' = c$ in frame S', which moves relative to frame S with speed v, then $u = c$ in frame S. This verifies Einstein's postulate about the constancy of the speed of light in all inertial frames.



$$\vec{u} = \frac{d\vec{r}}{dt} \qquad \vec{u}' = \frac{d\vec{r}'}{dt'}$$

Rapidity

$$\begin{cases} \dfrac{u}{c} \equiv \tanh \rho_u \\ \dfrac{u'}{c} \equiv \tanh \rho_{u'} \\ \dfrac{v}{c} \equiv \tanh \rho_v \end{cases}$$

For parallel velocities, we have

$$\tanh \rho_u = \frac{\tanh \rho_{u'} + \tanh \rho_v}{1 + \tanh \rho_v \tanh \rho_{u'}}$$

or, $\tanh \rho_u = \tanh(\rho_{u'} + \rho_v)$

$$\therefore \boxed{\rho_u = \rho_{u'} + \rho_v}$$

Thus, the utility of rapidity, ρ, as a variable is that it adds like the nonrelativistic velocity.

The Lorentz invariant quantity for the velocity 4-vector is

$$U_0^2 - \vec{U}^2 = \gamma^2 (c^2 - \vec{u}^2) = \gamma^2 c^2 (1 - \beta^2)$$
$$= c^2$$