Radiation by Moving Charges

The electromagnetic fields $F^{\alpha\beta}$ arising from an external source J^α satisfy the inhomogeneous Maxwell equations,

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

(see page 11-40)

The homogeneous Maxwell equations imply that the fields can be written in terms of the potentials:

$$\boxed{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha} \; .$$

Thus, we have

$$\partial_\alpha \partial^\alpha A^\beta - \partial^\beta (\partial_\alpha A^\alpha) = \frac{4\pi}{c} J^\beta$$

or,

$$\Box A^\beta - \partial^\beta (\partial_\alpha A^\alpha) = \frac{4\pi}{c} J^\beta \, .$$

If the potentials satisfy the Lorentz condition, $\partial_\alpha A^\alpha = 0$, then they are solutions of the four-dimensional wave equation,

$$\boxed{\Box A^\beta = \frac{4\pi}{c} J^\beta(x)} \; .$$

The solution can be accomplished by finding a Green function $D(x, x')$ for the equation,

$$\boxed{\Box_x D(x,x') = \delta_4(x-x')},$$

where $\delta_4(x-x') = \delta_1(x_0-x_0')\delta_3(\vec{r}-\vec{r}\,')$.

The solutions for the Green function were found in Chapter 6 (See page 6-22).

The <u>retarded or causal Green function</u> is

$$D_r(x-x') = \frac{1}{4\pi c}\,\frac{\delta_1\left(t-t'-\frac{R}{c}\right)}{R}$$

or,

$$D_r(x-x') = \frac{1}{4\pi R}\,\delta_1(x_0-x_0'-R)$$

where $R = |\vec{r}-\vec{r}\,'|$. Clearly $D_r(x-x') = 0$ unless

$t-t' = \frac{R}{c} \Rightarrow t-t' \geq 0$. Thus, we may write

$$\boxed{D_r(x-x') = \frac{\theta(x_0-x_0')}{4\pi R}\,\delta_1(x_0-x_0'-R)},$$

where $\theta(x_0-x_0') \equiv \begin{cases} 1, & \text{for } x_0-x_0' > 0 \\ 0, & \text{for } x_0-x_0' < 0 \end{cases}$

This Green function can be put in covariant form by using the identity,

$$\delta[(x-x')^2] = \delta[(x_0-x_0')^2 - |\vec{r}-\vec{r}$$

$$= \delta[(x_0-x_0'-R)(x_0-x_0'$$

$$= \frac{1}{2R}[\delta_1(x_0-x_0'-R) + \delta_1(x_0$$

Then, since the theta function select
of the two terms, we have

$$\boxed{D_r(x-x') = \frac{1}{2\pi}\Theta(x_0-x_0')\,\delta_1[(x-x'}$$

The theta function, when constrained b
delta function, is invariant under p
Lorentz transformations. If the so
are localized in space and time,
4-vector potential is

$$\boxed{A^\alpha(x) = \frac{4\pi}{c}\int d^4x'\, D_r(x-x')\,J^\alpha}$$

Consider a point charge $\underline{e}$ whose p
in the inertial frame S is $\vec{r}(t)$.
its charge density and current de
in that frame are

$$\begin{cases} \rho(\vec{x},t) = e\,\delta_3(\vec{x}-\vec{r}(t)) \\ \vec{J}(\vec{x},t) = e\,\vec{v}(t)\,\delta_3(\vec{x}-\vec{r}(t \end{cases}$$

where $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ is the charge's velocity in S. In frame S, the charge's <u>4-velocity</u> is

$$U^\alpha = (\gamma c, \gamma \vec{v}) \equiv (U_0, \vec{U})$$

and its coordinate 4-vector is

$$r^\alpha = (ct, \vec{r}(t)).$$

Clearly we can write*

$$\vec{J}(\vec{x}, t) = e \int d\tau \, \delta_1(t-\tau) \, \vec{v}(\tau) \, \delta_3(\vec{x} - \vec{r}(\tau))$$

where $\vec{v}(\tau) = \frac{d\vec{r}(\tau)}{d\tau}$.

Now $$\delta_1(t-\tau) = c\,\delta_1(ct - c\tau)$$
$$= c\,\delta_1(x_0 - c\tau)$$

and $\delta_1(t-\tau)\delta_3(\vec{x}-\vec{r}(\tau)) = c\,\delta_4(x - r(\tau))$.
Hence,

$$\vec{J}(x) = ec \int d\tau \, \vec{U}(\tau) \, \delta_4(x - r(\tau)) .$$

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* We choose τ as the charge's proper time.

and similarly for $J^0 = c\rho(x)$. Th
covariant form, we have

$$\boxed{J^\alpha(x) = ec\int d\tau \ U^\alpha(\tau)\,\delta_4(}$$

Now recall that

$$A^\alpha(x) = \frac{4\pi}{c}\int d^4x'\, D_r(x-x')\, J^\alpha$$

where here,

$$J^\alpha(x') = ec\int d\tau \ U^\alpha(\tau)\,\delta_4(x$$

Upon integration over d^4x', we obt

$$\boxed{A^\alpha(x) = 4\pi e\int d\tau \ U^\alpha(\tau)\, D_r[x}$$

or,

$$\boxed{A^\alpha(x) = 2e\int d\tau \ U^\alpha(\tau)\,\theta(x_0 - r_0(}$$

The only contribution to this integra
$\tau = \tau_0$, where

$$[x - r(\tau_0)]^2 \equiv 0 \qquad \Big(\begin{matrix}\text{lig}\\ \text{co}\end{matrix}$$

and $\quad x_0 - r_0(\tau_0) > 0 .$ (ret
con

To evaluate the integral for $A^\alpha(x)$,
the identity

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{\left|\left(\frac{df}{dx}\right)_{x=x_i}\right|}$$

where $f(x_i) = 0$. Now,

$$\frac{d}{d\tau}[x-r(\tau)]^2 = 2[x-r(\tau)] \cdot \frac{d}{d\tau}[$$

$$= -2\,U(\tau)\cdot[x-r($$

Hence,

$$\delta\left([x-r(\tau)]^2\right) = \frac{\delta(\tau-\tau_0)}{2\,U\cdot[x-r(\tau)]}$$

and

$$\boxed{A^\alpha(x) = \left.\frac{e\,U^\alpha(\tau)}{U\cdot[x-r(\tau)]}\right|_{\tau=}}$$

The potentials above are known as th
Liénard - Wiechert potentials.

We may write the potentials in noncovariant form as follows. The <u>light-cone condition</u> is

$$[x - r(\tau_0)]^2 = [x_0 - r_0(\tau_0)]^2 - [\vec{x} - \vec{r}(\tau_0)]^2 = 0$$

$$\Rightarrow \quad x_0 - r_0(\tau_0) = |\vec{x} - \vec{r}(\tau_0)| \equiv R \, .$$

Then $\quad U \cdot [x - r(\tau_0)] = U_0 [x_0 - r_0(\tau_0)] - \vec{U} \cdot [\vec{x} - \vec{r}(\tau_0)]$

$$= \gamma c R - \gamma \vec{v} \cdot \vec{R}$$

$$= \gamma c R (1 - \vec{\beta} \cdot \hat{n})$$

where $\vec{R} \equiv \vec{x} - \vec{r}(\tau_0) \equiv R \hat{n}$

and $\vec{\beta} = \dfrac{\vec{v}}{c}$.

Thus, the potentials $A^\alpha = (\Phi, \vec{A})$ may be written as

$$\boxed{\begin{aligned} \Phi(\vec{x}, t) &= \left[\frac{e}{(1 - \vec{\beta} \cdot \hat{n}) R} \right]_{ret} \\ \vec{A}(\vec{x}, t) &= \left[\frac{e \vec{\beta}}{(1 - \vec{\beta} \cdot \hat{n}) R} \right]_{ret} \end{aligned}}$$

where $[\ ]_{ret}$ means to evaluate the quantity in brackets at the retarded time,

$$x_0 - r_0(\tau_0) = R\,.$$

The corresponding fields are

$$\boxed{\begin{aligned} \vec{B} &= [\hat{n} \times \vec{E}]_{ret} \\ \vec{E}(\vec{x},t) &= e\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right]_{ret} \\ &\quad + \frac{e}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3R}\right]_{ret} \end{aligned}}$$

These fields divide naturally into "velocity fields," which are independent of the acceleration $\dot{\vec{\beta}}$, and "acceleration fields," which depend linearly on $\dot{\vec{\beta}}$.

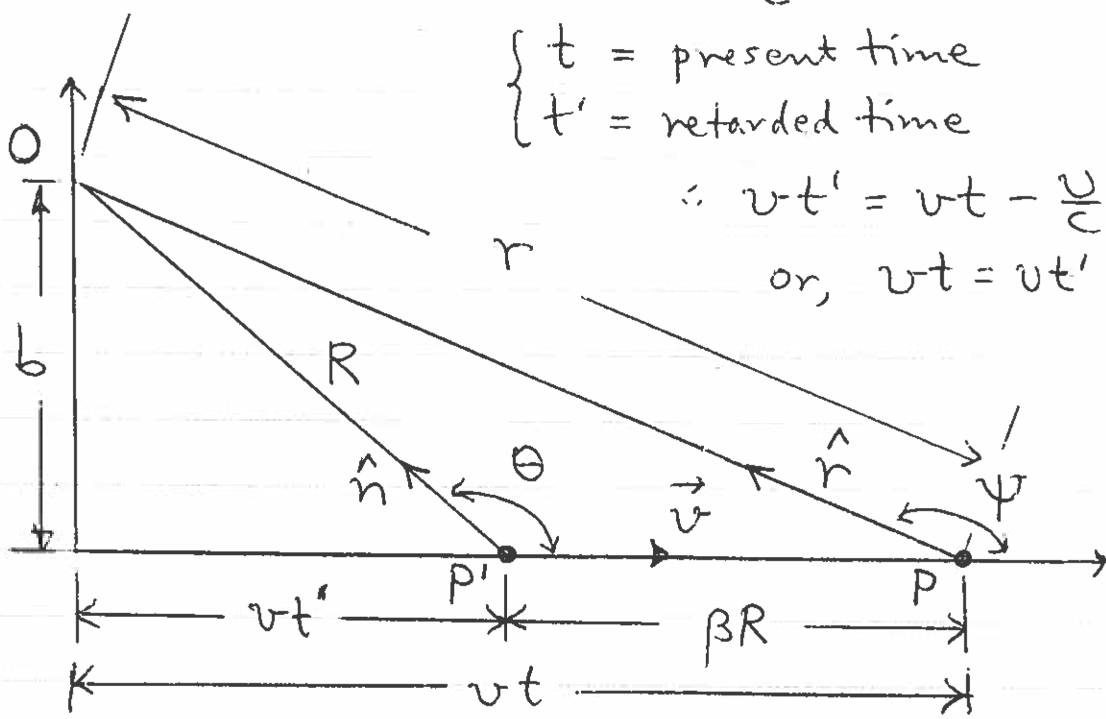
It may be worthwhile to rewrite the fields here, given in terms of the charge's retarded position, to an expression given in terms of the charge's present position, as discussed in Chapter 11 for a charge in uniform motion.

$$t' = t - \frac{R}{c}$$

$$\begin{cases} t = \text{present time} \\ t' = \text{retarded time} \end{cases}$$

$$\therefore \; vt' = vt - \frac{v}{c}R$$

$$\text{or,} \quad vt = vt' + \beta R$$



P = present position of charge
P' = retarded position of charge
O = observation point

Let $\vec{r} = r\hat{r}$.
Then, from the figure,

$$\vec{r} + \vec{\beta}R = \hat{n}R \equiv \vec{R}$$

$$\Rightarrow \quad \boxed{\hat{n} - \vec{\beta} = \frac{\vec{r}}{R}}$$

From the figure, we have

$$b = R\sin(\pi - \theta) = r\sin(\pi - \psi)$$

$$\text{or,} \quad b = R\sin\theta = r\sin\psi .$$

Multiply by β^2, then subtract from r^2.

Then $R^2(1-\cos^2\theta) = r^2 \sin^2\psi$

$$\Rightarrow \quad r^2 - \beta^2 R^2(1-\cos^2\theta) = r^2(1-\beta^2\sin^2\psi)$$

Now $\vec{r} = R(\hat{n}-\vec{\beta})$

$$\Rightarrow \quad r^2 = R^2(1+\beta^2 - 2\hat{n}\cdot\vec{\beta})$$

$$\therefore \quad R^2(1+\beta^2-2\hat{n}\cdot\vec{\beta}) - \beta^2 R^2(1-\cos^2\theta) = r^2(1-\beta^2\sin^2\psi)$$

$$R^2(1-2\hat{n}\cdot\vec{\beta} + \beta^2\cos^2\theta) = r^2(1-\beta^2\sin^2\psi)$$

But $\vec{\beta}\cdot\hat{n} = \beta\cos\theta$

$$\Rightarrow \quad \boxed{R^2(1-\vec{\beta}\cdot\hat{n})^2 = r^2(1-\beta^2\sin^2\psi)}$$

Hence, the velocity field can be written as

$$\underbrace{e\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3 R^2}\right]_{ret}}_{\vec{E}_{vel} \text{ in terms of retarded position}} = \underbrace{\frac{e\vec{r}}{r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}}_{\vec{E}_{vel} \text{ in terms of present position}},$$

in agreement with the result on page 11-51.

Now $\quad \vec{B} = \hat{n} \times \vec{E}$

where $\quad \hat{n} = \vec{\beta} + \frac{\vec{r}}{R}$

$\therefore \quad \hat{n} \times \vec{r} = \vec{\beta} \times \vec{r}$

and so

$$\vec{B}_v = \frac{e \quad \vec{\beta} \times \vec{r}}{r^3 \gamma^2 (1-\beta^2 \sin^2\psi)^{3/2}} .$$

At non-relativistic speeds,

$$\vec{B}_v \simeq \frac{e}{c} \frac{\vec{v} \times \vec{r}}{r^3} ,$$

which is just the Ampère-Biot-Savart expression for the magnetic field of a moving charge.

Total Power Radiated by an Accelerated
Larmor's Formula and its Relativistic

Suppose that a charge is accelerated b
observed in a reference frame wh

Then the acceleration field,

$$\vec{E}_a = \frac{e}{c}\left[\frac{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right]}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right]$$

reduces to

$$\boxed{\vec{E}_a \simeq \frac{e}{c}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R}\right]_{ret}}$$

The instantaneous energy flux is gi
the Poynting vector,

$$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}).$$

Now $\quad \vec{B}_a = (\hat{n}\times\vec{E}_a)_{ret}$

$$\Rightarrow \quad \vec{E}_a\times\vec{B}_a = \left[\vec{E}_a\times(\hat{n}\times\vec{E}_a)\right]_{ret}$$

$$= |\vec{E}_a|^2\,\hat{n} - \vec{E}_a(\vec{E}_a\cdot\hat{n})$$

Thus, $\vec{S} = \frac{c}{4\pi} |\vec{E}_a|^2 \hat{n}$.

Now $|\vec{S}| = \frac{dP}{R^2 d\Omega}$ is the power radiated per unit area.

Then the power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} |R\vec{E}_a|^2$$

or,

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} |\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2$$

We can write $\hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) = \hat{n}(\hat{n} \cdot \dot{\vec{\beta}}) - \dot{\vec{\beta}}$

$$\Rightarrow |\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2 = (\hat{n} \cdot \dot{\vec{\beta}})^2 + \dot{\vec{\beta}}^2 - 2(\hat{n} \cdot \dot{\vec{\beta}})^2$$

$$= \dot{\vec{\beta}}^2 - (\hat{n} \cdot \dot{\vec{\beta}})^2$$

$$\equiv \frac{\dot{\vec{v}}^2}{c^2} \sin^2 \Theta ,$$

where Θ is the angle between the acceleration $\dot{\vec{v}}$ and $\hat{n}$.

Hence,

$$\boxed{\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\dot{\vec{v}}|^2 \sin^2 \Theta}$$

The total instantaneous power radiated is obtained by integrating over all solid angle:

$$P = \frac{e^2}{4\pi c^3} |\dot{\vec{v}}|^2 \; 2\pi \int_0^\pi \sin^3\Theta \, d\Theta$$

Aside: Let $x = \cos\Theta$

$dx = -\sin\Theta \, d\Theta$.

Then $\int_0^\pi \sin^3\Theta \, d\Theta = \int_{-1}^{1} (1-x^2)\, dx = \frac{4}{3}$

$$\therefore \quad \boxed{P = \frac{2}{3}\frac{e^2}{c^3} |\dot{\vec{v}}|^2}$$

This is the Larmor formula for a nonrelativistic accelerated charge.

Let us now generalize Larmor's formula to yield a result that is valid for arbitrary velocities of the charge. The desired generalization for the power is a Lorentz invariant that reduces to the Larmor formula for $\beta \ll 1$.