Nonrelativistically for a particle with mass $\underline{m}$ and charge $\underline{e}$, we can write the Larmor formula in the suggestive form,

$$\underline{P} = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}\right),$$

where $\vec{p} = m\vec{v}$ is the nonrelativistic momentum. The Lorentz invariant generalization is

$$\boxed{\underline{P} = \frac{-2}{3}\frac{e^2}{m^2c^3}\left(\frac{dP^\mu}{d\tau}\cdot\frac{dP_\mu}{d\tau}\right)}$$

where $d\tau = dt/\gamma$ is the proper time element,

Now $\qquad p^\mu = (E/c, \vec{p})$

$$\Rightarrow \quad \frac{dp^\mu}{dt} = \left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$$

and $\quad \dfrac{dp^\mu}{d\tau} = \gamma\left(\dfrac{1}{c}\dfrac{dE}{dt}, \dfrac{d\vec{p}}{dt}\right)$.

$$\Rightarrow \quad \frac{dp^\mu}{d\tau}\cdot\frac{dP_\mu}{d\tau} = \gamma^2\left[\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2 - \left(\frac{d\vec{p}}{dt}\right)^2\right]$$

We have $\begin{cases} E = mc^2\gamma \\ \vec{p} = m\gamma\vec{v} = mc\gamma\vec{\beta} \end{cases}$

where $\gamma = (1 - \vec{\beta}\cdot\vec{\beta})^{-\frac{1}{2}}$.

$$\Rightarrow \frac{d\gamma}{dt} = \gamma^3 (\vec{\beta}\cdot\dot{\vec{\beta}})$$

so that $$\frac{dE}{dt} = mc^2\gamma^3(\vec{\beta}\cdot\dot{\vec{\beta}}$$

$$\frac{d\vec{p}}{dt} = mc\vec{\beta}\frac{d\gamma}{dt} + mc\gamma\dot{\vec{\beta}}$$

$$\frac{d\vec{p}}{dt} = mc\gamma^3\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}}) + mc\gamma\dot{\vec{\beta}}$$

$$\frac{dp^\mu}{d\tau}\cdot\frac{dp_\mu}{d\tau} = m^2c^2\gamma^2\Big[\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2$$

$$-\Big\{\gamma^3\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}}) + \gamma\dot{\vec{\beta}}$$

$$\frac{dp^\mu}{d\tau}\cdot\frac{dp_\mu}{d\tau} = m^2c^2\gamma^2\Big[\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2$$

$$-\gamma^6\beta^2(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \gamma^2\dot{\vec{\beta}}$$

$$-2\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2\Big]$$

Now $\gamma^6 - \gamma^6\beta^2 = \gamma^6(1-\beta^2) = \gamma$

$$\Rightarrow \gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \gamma^6\beta^2(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - 2\gamma^4$$
$$= -\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2$$

$$\therefore \quad \frac{dP^\mu}{d\tau} \cdot \frac{dP_\mu}{d\tau} = -m^2c^2\gamma^2\Big[\gamma^4 (\beta$$

$$+ \gamma^2 \dot{\vec{\beta}}$$

$$\Rightarrow \underline{P} = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[(\vec{\beta}\cdot\dot{\vec{\beta}})^2 + \frac{\dot{\vec{\beta}}^2}{\gamma^2}\right.$$

or $\underline{P} = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[(\vec{\beta}\cdot\dot{\vec{\beta}})^2 + (1-\right.$

$$\underline{P} = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\vec{\beta}}^2 - \vec{\beta}^2\dot{\vec{\beta}}^2 + (\vec{\beta}\cdot\right.$$

Now $(\vec{a}\times\vec{b})^2 = \vec{a}^2\,\vec{b}^2 - (\vec{a}\cdot\vec{b})$

$$\Rightarrow \boxed{\underline{P} = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})\right.}$$

This is the Liénard result (1898).

Thomson Scattering of Radiation

Recall that the power radiated per unit solid angle by a particle of charge e in non-relativistic motion is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2$$

or,
$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} \left| \hat{n} \times (\hat{n} \times \dot{\vec{v}}) \right|^2 .$$

We consider the situation where a plane wave of monochromatic electromagnetic radiation is incident on a free particle of charge e and mass m. The particle as a consequence will be accelerated, and so emit radiation. This radiation will be emitted in directions other than that of the incident plane wave, but for nonrelativistic motion of the particle, it will have the same frequency as the incident radiation.

According to the equation above, the power radiated into polarization state $\hat{\epsilon}$ is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} \left| \hat{\epsilon}^* \cdot \dot{\vec{v}} \right|^2 .$$

Aside

Recall for $v \ll c$,

$$\vec{E}_a \simeq \frac{e}{c}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R}\right]_{ret}$$

Then $\hat{n}\cdot\vec{E}_a = 0$.

Now, $\hat{n}\times(\hat{n}\times\dot{\vec{\beta}}) = \hat{n}(\hat{n}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}}$

If $\vec{E}_a = \hat{\epsilon}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

then, $\hat{\epsilon}^*\cdot[\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})] = -\hat{\epsilon}^*\cdot\dot{\vec{\beta}}$

$$\therefore \left|\hat{\epsilon}^*\cdot[\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})]\right| = \left|\hat{\epsilon}^*\cdot\dot{\vec{\beta}}\right|$$

Let the incident plane wave have propagation vector $\vec{k}_0$ and polarization vector $\hat{\epsilon}_0$. Then the electric field can be written in complex form as

$$\vec{E}(\vec{r},t) = \hat{\epsilon}_0 E_0 e^{i(\vec{k}_0 \cdot \vec{r} - \omega t)}.$$

For nonrelativistic motion, the force on the charge is

$$m\dot{\vec{v}}(t) = e\vec{E}(\vec{r},t)$$

so, $$\dot{\vec{v}}(t) = \hat{\epsilon}_0 \frac{e}{m} E_0 e^{i(\vec{k}_0 \cdot \vec{r} - \omega t)}.$$

If we assume that the charge moves a negligible part of a wavelength during one cycle of oscillation, then the time average of $|\dot{\vec{v}}|^2$ is $\frac{1}{2}\mathrm{Re}(\dot{\vec{v}} \cdot \dot{\vec{v}}^*)$. Hence, the time-averaged power radiated per unit solid angle is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{2} \frac{e^2}{4\pi c^3} \left(\frac{e}{m}\right)^2 |E_0|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2$$

or, $$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} \left(\frac{e^2}{mc^2}\right)^2 |E_0|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2}$$

The differential cross section for scattering of radiation by a free charge is defined by

$$\frac{d\sigma}{d\Omega} = \frac{\text{Energy radiated per unit time per unit solid angle}}{\text{Incident energy flux per unit time per unit area}}$$

The incident energy flux is just the time-averaged Poynting vector for the plane wave, i.e.,

$$\frac{c}{8\pi}|E_0|^2.$$

Thus the differential cross section is

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2}$$

Recall that the acceleration field is

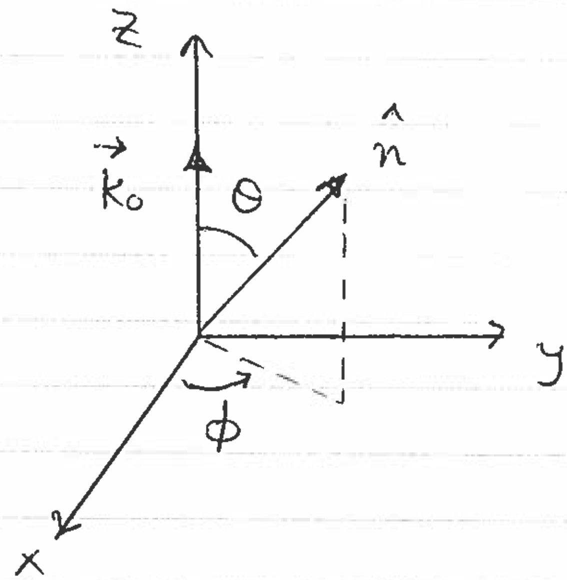
$$\vec{E}_a = \frac{e}{c}\left[\frac{\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})}{R}\right]_{ret},$$

so that the polarization vector $\hat{\epsilon}$ of the outgoing wave is perpendicular to $\hat{n}$, which is along the direction of observation.

Suppose $\vec{K}_0 \equiv K_0 \hat{z}$ is the propagation vector for the incident wave.

Let θ denote the angle between $\vec{k}_o$ and $\hat{n}$:

$$\hat{k}_o \cdot \hat{n} = \cos\theta$$



θ = scattering angle

Then $\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

Now $\hat{n} \equiv \hat{r}$

and $\hat{\theta} = \dfrac{\partial\hat{r}}{\partial\theta} \equiv \hat{\epsilon}_\theta$

$$\Rightarrow \hat{\epsilon}_\theta = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$

Also, $\hat{\rho} \equiv \hat{x}\cos\phi + \hat{y}\sin\phi$

and $\hat{\phi} = \dfrac{\partial\hat{\rho}}{\partial\phi} \equiv \hat{\epsilon}_\phi$

$$\Rightarrow \hat{\epsilon}_\phi = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

For an incident linearly polarized wave with $\hat{\epsilon}_0 = \hat{x}$, the angular distribution summed over final polarizations $\hat{\epsilon} = \hat{\epsilon}_\theta$ and $\hat{\epsilon} = \hat{\epsilon}_\phi$ is

$$|\hat{x}\cdot\hat{\epsilon}_\theta|^2 + |\hat{x}\cdot\hat{\epsilon}_\phi|^2 = \cos^2\theta\cos^2\phi + \sin^2\phi .$$

Similarly, for an incident polarized wave with $\hat{\epsilon}_0 = \hat{y}$, the angular distribution summed over final polarizations $\hat{\epsilon} = \hat{\epsilon}_\theta$ and $\hat{\epsilon} = \hat{\epsilon}_\phi$ is

$$|\hat{y}\cdot\hat{\epsilon}_\theta|^2 + |\hat{y}\cdot\hat{\epsilon}_\phi|^2 = \cos^2\theta\sin^2\phi + \cos^2\phi ,$$

For unpolarized incident radiation, the angular distribution averaged over initial polarizations $\hat{\epsilon}_0 = \hat{x}$ and $\hat{\epsilon}_0 = \hat{y}$ and summed over final polarizations $\hat{\epsilon} = \hat{\epsilon}_\theta$ and $\hat{\epsilon} = \hat{\epsilon}_\phi$ is

$$\frac{1}{2}\left(\cos^2\theta\cos^2\phi + \sin^2\phi\right) + \frac{1}{2}\left(\cos^2\theta\sin^2\phi + \cos^2\phi\right) = \frac{1}{2}\left(1+\cos^2\theta\right).$$

Hence, for <u>unpolarized incident radiation</u>, the scattering cross section is

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \cdot \frac{1}{2}\left(1+\cos^2\theta\right)}$$

This is called the <u>Thomson formula</u> for scattering of radiation by a free charge.

* Thomson scattering is the low-energy limit of Compton scattering, as described by classical e&m. (14-22)

The total scattering cross section, called the Thomson cross section, is given by

$$\sigma_T = \left(\frac{e^2}{mc^2}\right)^2 \frac{1}{2} \int_0^{2\pi}\int_0^{\pi} (1+\cos^2\theta)\sin\theta\, d\theta\, d\phi$$

$$\Rightarrow \sigma_T = \pi\left(\frac{e^2}{mc^2}\right)^2 \int_{-1}^{1} (1+\xi^2)\, d\xi$$

where $\xi \equiv \cos\theta$.

Since $\int_{-1}^{1}(1+\xi^2)\,d\xi = 2\left(\xi+\frac{\xi^3}{3}\right)\Big|_0^1 = \frac{8}{3}$,

we have $\boxed{\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2}$.

For electrons, $\frac{e^2}{mc^2} \equiv r_e = 2,818 \times 10^{-13}$ cm,

is called the classical electron radius.

The classical Thomson formula is valid only at low frequencies where the momentum of the incident photon can be ignored. When the photon's momentum $\hbar\omega/c$ becomes comparable to or larger than mc, modifications occur. The most important one is kinematical: the energy or momentum of the scattered photon is less than the incident energy because the charged particle recoils during the collision. A quantum-mechanical calculation of the scattering of photons by spinless point particles of charge e and mass m gives

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{k'}{k}\right)^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2},$$

where the ratio of the outgoing to the incident wave number is given by the Compton formula;

$$\boxed{\frac{k'}{k} = \frac{1}{1 + \frac{\hbar\omega}{mc^2}(1-\cos\theta)}},$$

where θ is the scattering angle in the laboratory (the rest frame of the target).