Candidacy Exam Spring 2003

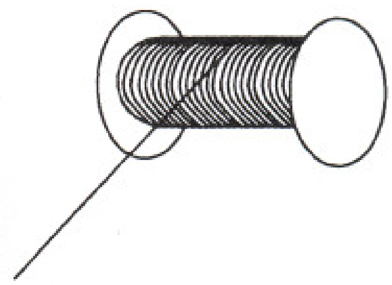
Classical Mechanics

Do all problems

(1) (20 pts.)

Consider a tightly wound spool of thread lying on a table, as shown at right. With the spool initially at rest, you start to pull the loose thread towards you. Suppose that there is so much friction between the spool and the table that the spool rolls without slipping.

In which direction does the spool start to move? Towards you or away from you? Explain your reasoning.



(2) (40 pts) Dust particles in the solar system feel a repulsive force $\boldsymbol{F}_R(\mathrm{r})$ due to the radiation pressure from the sun, proportional to the cross-sectional area of the particle, as well as the usual gravitational attraction. The radiation force on a particle with radius a at a distance r from the sun is

$$\mathbf{F}_R(r) = K\frac{\pi a^2}{r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction and K depends on the radiant flux from the sun and on the properties of the particle.

(a) Write down the Lagrangian for this system (sun + 1 dust particle) in terms of the particle radius a and the particle mass density ρ, the mass of the sun M, and the gravitational constant G. You can consider the sun as a fixed point at the origin of your system. Why can you argue that this system is effectively two-dimensional?

(b) Find the canonical momenta and write down the Hamiltonian for this system.

(c) Write down Hamilton's equations of motion for this system. List all conserved quantities.

(d) Use this list to show that the Hamiltonian and the equation of motion for r can be given in terms of an effective 1-dimensional potential $V(r)$; find $V(r)$.

(e) Show that there is a critical radius a_0 such that smaller particles will be driven out of the solar system while larger particles will show (qualitatively) normal gravitational behavior. Find a_0.

(f) Sketch (roughly, but indicating proper asymptotic behavior and the presence of any maxima or minima) the effective potential V(r) for $a=a_0/2$ and for $a=2a_0$ and some given angular momentum ℓ. On these sketches, indicate the regimes for different sorts of motion (bound, free, circular....)

(3; 40 pts) Cat toy, shown at right: Three small objects (stuffed mice), each with mass m, are attached to each other with springs, each with spring constant k. At rest, the springs are at their equilibrium extension. The ring, with radius R, is itself fixed. As a first approximation to explore the behavior of this toy, assume that there is no friction with respect to the ring: the object/spring system can freely rotate within the ring. For this simple system, answer the following questions:

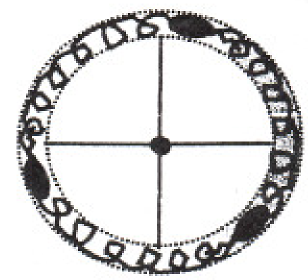


Figure 2. Cross-section of a cat toy. There is a cover with slits over this, so that the small objects within can only be seen when they are at the right position. Ignore the finite width of the ring.

(a; 15 pts) Find the equations of motion for the stuffed mice.

(b; 15 pts) Solve for the eigenfrequencies and the normal modes of this problem.

(c; 10 pts) Now consider how the cat will actually play with this toy: She will bat one of the objects. Assume that the mice begin with 0 displacement from their equilibrium, and that the cat simply starts one of the mice moving. Describe the subsequent motion in terms of the normal modes you have found.