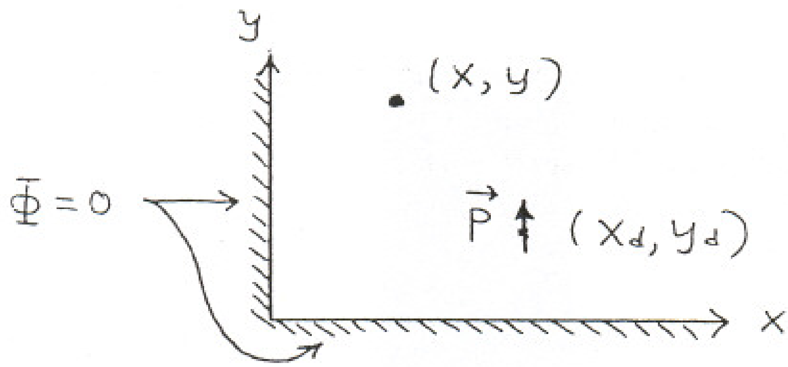
Candidacy Exam Spring 2000

Electricity and Magnetism

Instructions: Work any three of the following four problems. Each problem will be graded for equal credit.

1. An electric dipole $\mathbf{p} = p\hat{y}$ is placed in free space at position (x_d, y_d). The space surrounding the dipole is bounded by a grounded conductor as shown in the figure. Use the method of images to derive the potential $\Phi(x, y)$ produced by the dipole at point (x, y). Why does this method work?



2. (a) State Maxwell's equations in free space.

   (*b*) Derive from these equations the equation of continuity for the charge and current densities.

   (*c*) Introduce the scalar and vector potentials and define the Coulomb gauge.

   (*d*) Derive the wave equation for the vector potential in the Coulomb gauge. Express the source as a constant times $\mathbf{J}_T$, the transverse current density, and express $\mathbf{J}_L$, the longitudinal current density, in terms of the scalar potential. (Note that the total current density $\mathbf{J}$ is the sum of $\mathbf{J}_L$ and $\mathbf{J}_T$.)

3. A magnetic dipole $\mathbf{m} = -m\hat{z}$ is located at the origin in an otherwise uniform magnetic field $\mathbf{B} = B\hat{z}$. Show that there exists a sphere, centered at the origin, through which no magnetic field lines pass. Find the radius of this sphere.

4. The nonrelativistic electric field $\mathbf{E}$ in the radiation zone of an accelerated charge e is given in SI units by

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0 c^2}\frac{1}{r}\left[\hat{r} \times (\hat{r} \times \mathbf{a})\right],$$

where $\hat{r}$ is a unit vector in the radial direction and $\mathbf{a}$ is the particle's acceleration. Write down the Poynting vector and compute the total radiated power.

Possibly useful information:

$$\int_0^\pi \sin n\theta \, \sin m\theta \, d\theta = \frac{\pi}{2}\,\delta_{nm}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$
$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Cylindrical coordinates:

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical coordinates:

$$\nabla \times \mathbf{A} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$