Candidacy Exam, Spring 2003
Electricity and Magnetism

1. (25 points) Laplace's Equation

   A grounded metal sphere of radius R is located at the origin. It is covered with a dielectric layer that is also of thickness R and permitivity ϵ. A uniform external field, $E_0\hat{z}$, is applied in the z-direction.

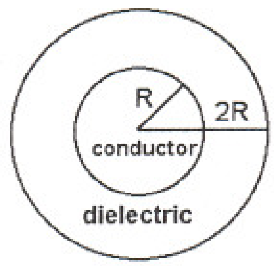

Figure 1: Dielectric covered conducting sphere

Solutions to Laplace's equation with azimuthal symmetry are expressed as an expansion in Legendre polynomials,

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + B_l \frac{P_l(\cos\theta)}{r^{l+1}} . \quad (1)$$

Find the potential inside the dielectric and in the vacuum outside the sphere.
Hint 1: The external field produces a potential,

$$V^{ex}(r,\theta) = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta) .$$

Hint 2: You will get most of the credit for setting the problem up correctly rather than working through all the algebra. You may wish to start by identifying the correct boundary conditions.

2. (20 points) Gauss's Law

A slab of uniform charge density, ρ_0, with thickness h and of infinite lateral extent is located in the x-y plane. The slab is centered so that it occupies the region $-h/2 < z < h/2$.

Now, in addition to the slab, two grounded conducting plates are added so that one is located at $z = a$ above the slab and one is located at $z = -b$ below the slab. Both a and b are greater than the thickness of the slab.

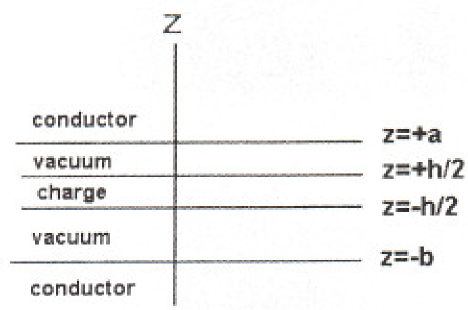

Figure 2: Charge density between grounded conductors

One immediately knows that there are three charge densities which create the electric field. They are the original charge density ρ_0 and two surface charge densities σ_a, σ_b that must appear on the bounding conducting plates.

What is the electric field in the three regions of space between the two plates ?

$$E_z^I(-b < z < -h/2) = ? \quad (2)$$

$$E_z^{II}(-h/2 < z < h/2) = ? \quad (3)$$

$$E_z^{III}(h/2 < z < a) = ? \quad (4)$$

(Hint: Do not forget that the conductors are grounded.)

3. (30 points) Induction

A long solenoid carries a current that oscillates so that the magnetic field inside the solenoid is essentially uniform and varies sinusoidally in time,

$$\vec{B}(t) = \hat{z} B_0 \sin\left(2\pi \frac{t}{\tau}\right), \tag{5}$$

where τ is a constant.

We will idealize this problem and assume that the time constant, τ, is large compared the time it would take light to travel the length of the solenoid, so there are no radiation effects, just inductive effects. The solenoid is of radius R and its axis of symmetry coincides with the z-axis. The length, L, is much greater than its radius. Assume that the current in the solenoid has been on a sufficiently long time in the past that any transients can be ignored.

According to Faraday's law with a circle around the z-axis of radius $s < R$, an electric field is generated inside the solenoid,

$$\begin{aligned}
\int \vec{E} \cdot d\vec{l} &= -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}, \\
2\pi s E_\phi(s) &= -\frac{\partial}{\partial t} B_z(t) \pi s^2, \\
\vec{E}(\vec{r}) &= -\frac{B_0 s \pi}{\tau} \cos\left(2\pi \frac{t}{\tau}\right) \hat{\phi}.
\end{aligned} \tag{6}$$

Now, the experiment is repeated, but this time the solenoid is half filled with a conducting material that occupies the region $y < 0$ inside the solenoid as shown in Fig. 3.

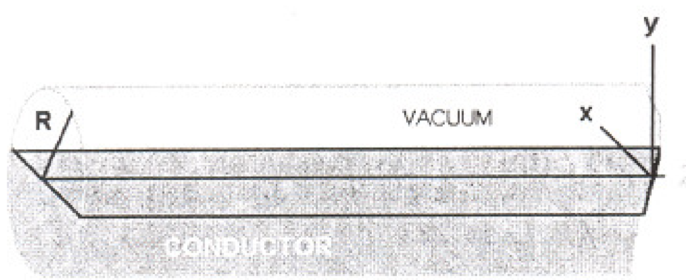

Figure 3: Solenoid half filled with conductor

The conductor may be taken as ideal, having no resistance. The current in the solenoid is the same as used to produce the original field, i.e.

$$\vec{K}(t) = \frac{\hat{\phi} B_0}{\mu_0} \sin\left(2\pi \frac{t}{\tau}\right) . \tag{7}$$

(a) (10 points) Show that the current, Eq. 7, results in the same electric field (Eq. 6) and the same magnetic field (Eq. 5) inside the vacuum region of the solenoid near its center in the presence of the conductor as it did without the conductor. (Of course, both must be zero inside the conductor.)

(b) (10 points) What is the surface charge density on the conductor on the x-z plane?

(c) (10 points) What is the surface current density, $\vec{K}(x,t)$, on the conductor on the x-z plane?

4. Magnetic Force and Energy (25 points)

(a) (10 pts) A cylindrical bar magnet has a permanent uniform magnetization density, $\vec{M} = M_0\hat{z}$ along its axis. The axis of the cylinder is the z-axis. Assume the length L of the cylinder is much greater than the radius, $L >> R$.

What is the magnetic field, $\vec{B}$, just outside of the end of the magnet near its center? (Hint: One possible solution employs the magnetization charge density, σ_m.)

(b) (10 points) If two such magnets are placed end-to-end, so that the north pole of one is near the south pole of the other and a small separation, h exists between them, then the field in the space will be just twice what the field was in part 4a as long as we can ignore edge effects.

What is the force of attraction between the two magnets?

(c) (5 points) If one of the magnets of part 4b is reversed, so that the two north poles are facing, then the field in the space is approximately zero, as long we can ignore edge effects.

What is the force of repulsion between the two magnets? (No calculation required.)

Would your method, applied to part 4b, give the correct answer here?