Fall 2001

QUANTUM MECHANICS

1) a)(5pts) Show that the eigenvalues of a Hermitian operator are real.

b)(5pts) Show that the eigenvectors of a Hermitian operator belonging to distinct eigenvalues are orthogonal.

Suppose that you have three particles, one in state $\Psi_a(x)$, one in $\Psi_b(x)$ and one in $\Psi_c(x)$. Assuming that these are orthonormal states construct the three-particle states (ignore the spin part) representing

c)(5pts) distinguishable particles

d)(5pts) identical bosons

e)(10pts) identical fermions.

2)(20pts) What is the probability that an electron in the ground state of hydrogen ($\Psi_{100} = (1/\sqrt{\pi a^3})e^{-r/a}$) will be found **inside** the nucleus? Let b be the radius of the nucleus.

a)(10pts) First calculate the exact answer assuming the wave function is correct all the way down to $r = 0$. Then, expand your result as a power series of $\epsilon = 2b/a$ and find the lowest order term.

b)(10pts) Alternatively you may assume that the wave function is constant over the tiny volume of the nucleus ($\Psi(0\langle r\langle b) = \Psi(0)$). Check that you get the same answer as in a) above. If $b = 10^{-15}\,\mathrm{m}$ and $a = 0.5\cdot 10^{-10}\,\mathrm{m}$ get a numerical estimate of this probability. Note that this also represents the "fraction of its time that the electron spends inside the nucleus".

3)(25pts) Imagine a system in which there are just two linearly independent states:
$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The most general state is a normalized linear combination: $|\Psi\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, with $|a|^2 + |b|^2 = 1$.

Suppose the (time independent) Hamiltonian is $\mathbf{H}= \begin{pmatrix} k & g \\ g & k \end{pmatrix}$, where k and g are real constants.

a) (10pts) Find the eigenvalues and (normalized) eigenvectors of this Hamiltonian.

b) (15pts) Suppose the system starts out (at $t = 0$) in state $|1\rangle$. What is the state at time t?

4)(25pts) Two identical bosons are placed inside an one-dimensional infinite square well of width a. They interact weakly with one another, via the potential

$$H' = V(x_1, x_2) = -aV_0\delta(x_1 - x_2)$$

(where V_0 is a constant and a is the width of the well).

a)(10pts) Find, ignoring the interaction between the particles, the ground state **and** first excited state of the system. Find both the wave functions and the associated energies.

b)(15pts) Use first-order perturbation theory to calculate the effect of the particle-particle interaction on the ground **and** first excited state energies.

Note: The single particle states for the well are $\Psi_n(x) = (\sqrt{2/a})\sin(n\pi x/a)$ with energy $E_n = n^2 E_1 = n^2\pi^2\hbar^2/2ma^2$

$$\int_0^\pi \sin^n(y)dy = \pi\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, \quad n = \text{even}$$