Fall 2009

QUANTUM MECHANICS

1)(20pts) The following questions require only short but clear answers, not lengthy calculations.

a) (5pts) Show that the matrix S that diagonalizes a Hermitian matrix H by a similarity transform ($H_D = SHS^{-1}$) is unitary.

b) (5pts) A Hermitian operator A is said to be positive-definite if, for any vector $|u\rangle$, $\langle u|A|u\rangle \geq 0$. Show that the operator $A = |a\rangle\langle a|$ is Hermitian and positive-definite.

c) (5pts) Write down an expression for the energy spectrum of a two-dimensional (2-D) isotropic (i.e. $k_x = k_y$) harmonic oscillator. What is the degree of degeneracy of the first three states?

d) (5pts) What are the possible total spin values when combining three spin one-half ($s = 1/2$) particles?

2)(20pts) For stationary states the mean value of the scalar product $\vec{r} \cdot \vec{p}$ has to be time-independent, i.e. $d\langle \vec{r} \cdot \vec{p}\rangle/dt = 0$. Using this fact derive the 'quantum virial theorem':

$$2\langle T\rangle = \langle \vec{r} \cdot \vec{\nabla} V(\vec{r})\rangle$$

where V is the potential energy and T is the kinetic energy operator of the particle. (You may use the relation $i\hbar\, d\langle A\rangle/dt = \langle [A, H]\rangle$, where A is any operator.)

3)(30pts) The following Hamiltonian represents a three-level system:

$$\mathbf{H} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix},$$ where a,b and c are real numbers.

a) (10pts) Find the eigenvalues and (normalized) eigenstates of $\mathbf{H}$ for the non-degenerate energies case.

b) (10pts) If the system starts out in the state: $|\Psi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, what is $|\Psi(t)\rangle$?

c) (10pts) If we perform a measurement of the energy in the state $|\Psi(0)\rangle$ (i.e. at time $t = 0$) what values might you get, and what is the probability of each?

4)(30pts) Consider the perturbation potential H' ($H' = \lambda r$ with $\lambda > 0$) applied to a Hydrogen atom. You should ignore spin-orbit, hyperfine, relativistic, etc. corrections.

a) (10pts) What is the perturbed energy to 1st order in the $n = 1$, $l = 0$ state?

b) (15pts) What are the perturbed energies in the $n = 2$, $l = 0, 1$ manifold of states? Explain clearly whether you must use degenerate perturbation theory or not.

c) (5pts) Is H' a valid perturbation potential to treat in 1st order theory? Explain qualitatively. (Hint: Consider behavior at small and large values of λr)

TABLE 4.7: The first few radial wave functions for hydrogen, $R_{nl}(r)$.

$R_{10} = 2a^{-3/2}\exp(-r/a)$
$R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)\exp(-r/2a)$ $R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\frac{r}{a}\exp(-r/2a)$

TABLE 4.3: The first few spherical harmonics, $Y_l^m(\theta, \phi)$.

$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$	$Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$
$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$	$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2}(5\cos^3\theta - 3\cos\theta)$
$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$	$Y_3^{\pm 1} = \mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$
$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1)$	$Y_3^{\pm 2} = \left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{\pm 2i\phi}$
$Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi}$	$Y_3^{\pm 3} = \mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta e^{\pm 3i\phi}$

The normalized angular wave functions[8] are called **spherical harmonics**:

$$Y_l^m(\theta, \phi) = \epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\, e^{im\phi}\, P_l^m(\cos\theta), \qquad [4.32]$$

where $\epsilon = (-1)^m$ for $m \geq 0$ and $\epsilon = 1$ for $m \leq 0$. As we shall prove later on, they are automatically orthogonal, so

$$\int_0^{2\pi}\int_0^{\pi}[Y_l^m(\theta, \phi)]^*[Y_{l'}^{m'}(\theta, \phi)]\sin\theta\, d\theta\, d\phi = \delta_{ll'}\delta_{mm'}, \qquad [4.33]$$

Integrals:

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Exponential integrals:

$$\int_0^{\infty} x^n e^{-x/a}\,dx = n!\,a^{n+1}$$

Gaussian integrals:

$$\int_0^{\infty} x^{2n} e^{-x^2/a^2}\,dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$$

$$\int_0^{\infty} x^{2n+1} e^{-x^2/a^2}\,dx = \frac{n!}{2}a^{2n+2}$$