Quantum and Atomic Physics

Work all problems. Parts (b), (c), and (d) do not necessarily rely on earlier parts, so if you cannot solve a sub-problem, move to the next sub-problem.

(40 pts.) 1. Suppose that the one-dimensional Schrödinger equation has been solved for a particle of mass m in the potential

$$V(x) = \begin{cases} \infty & \text{if } x < 0 \\ -\frac{C}{x} + \frac{D}{x^2} & \text{if } x > 0 \end{cases}$$

(where C and D are positive real constants of appropriate dimensions), and a bound state of the system has been found at

$$E_B = -\frac{3C^2}{16D}.$$

(15 pts.) a) Sketch the bound-state wave function for two cases: (1) if E_B is the ground-state energy, (2) if E_B is the energy of the first excited state. (Clearly display the steps that lead to your drawing and their justification.)

(5 pts.) b) Estimate the kinetic energy of the particle in the ground state with eigenenergy E_B.

(10 pts.) c) How would you approximate the potential to describe small-amplitude motion around the classical equilibrium? What are the energy eigenvalues in this approximation?

(10 pts.) d) Consider the potential

$$V(r) = -\frac{C}{r} + \frac{D}{r^2}$$

in three dimensions. Discuss the eigenfunctions in spherical polar coordinates. Describe the angle-dependence of the eigenfunctions.

NOTE: This problem can be solved exactly and E_B is not an exact energy eigenvalue. It serves here as illustration.

(35 pts.) 2. An electron is placed in a homogeneous magnetic field. Assume that the electron is fixed at a certain location, and its spin is the only degree of freedom in the problem. At time t=0 the spin-state of the electron can be written as

$$|t = 0> = e^{i\gamma} \cos\frac{\theta}{2} |\uparrow> + e^{i\delta} \sin\frac{\theta}{2} |\downarrow>$$

where $|\uparrow>$ and $|\downarrow>$ denote the orthonormal eigenstates of the z-component of the spin-operator, $\hat{S}_z$, and γ,δ,θ are real constants.

(5 pts.) a) Why is the $|t = 0>$ spin-state parametrized as above? Can one use fewer constants? (If no, why, if yes, how?)

(15 pts.) b) Calculate the time-dependent spin-state, $|t>$.

(15 pts.) c) Determine the expectation value of the components of the spin as a function of time.

NOTE: the Pauli spin matrices $\underline{\hat{S}} = \frac{\hbar}{2}\underline{\hat{\sigma}}$,

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

can be used.

(25 pts.) 3. A physical observable has the matrix representation

$$\underline{\underline{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(10 pts.) a) Find the eigenvalues and normalized eigenvectors of this observable.

(10 pts.) b) Calculate the matrix of the unitary transformation that brings $\underline{\underline{A}}$ into diagonal form.

(5 pts.) c) Give an example of a physical quantity which may be represented by $\underline{\underline{A}}$.