Quantum Mechanics

(30 pts.) 1. Answer the following:

(10 pts.) (a) For the angular momentum operator $\vec{J} = \hat{x} J_1 + \hat{y} J_2 + \hat{z} J_3$, derive the commutation relation that signifies that $J_1 + iJ_2$ is a raising operator.

(10 pts.) (b) Use the result of part (a), along with $[J^2, J_1 + iJ_2] = 0$, to demonstrate what quantum numbers the state $(J_1 + iJ_2)|j,m>$ has.

(10 pts.) (c) For a spinless non-relativistic particle of charge q moving in a uniform magnetic field $\vec{B}$, the Hamiltonian is $H = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2$ where one may choose $\vec{A}(\vec{r}) = \frac{1}{2}\vec{B} \times \vec{r}$ which satisfies $\vec{\nabla} \cdot \vec{A} = 0$. Derive the formula for the magnetic moment of the particle. (Ignore effects that are second-order in $\vec{B}$.) If the particle also has an intrinsic spin, state (do not derive) what the corrected expression for the magnetic moment is.

(30 pts.) 2. Consider an electron in a constant, uniform magnetic field $\vec{B}$ where the electron spin is the only relevant degree of freedom. In a basis of eigenstates of the component of spin parallel to $\vec{B}$ the $t = 0$ state of the electron is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(15 pts.) (a) Find the state of the electron at any $t > 0$.

(15 pts.) (b) The cyclotron frequency for a circular orbit of a classical charged particle in a uniform magnetic field is $\Omega = qB/mc$. Derive and discuss quantum mechanical electron spin precession of both the state and the spin eigenvalue, paying particular attention to the precession frequency and its relation to Ω.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(40 pts.) 3. A hydrogen atom is placed in a constant uniform electric field $\vec{\varepsilon} = \varepsilon \hat{z}$. Ignore spin and consider the effect of the resulting potential $V = -q\varepsilon z$ on the electron to first order in perturbation theory.

(15 pts.) (a) The energy shift of the $n = 1$ level can be obtained very simply. What is it? Show your reasoning.

(25 pts.) (b) The $n = 2$ level is degenerate and the first order energy shifts are the eigenvalues of the matrix formed by V in the unperturbed degenerate basis. Evaluate the shifts of the $n = 2$ level. {In determining which of the basis states are linked by non-zero matrix elements of V, it is helpful to consider symmetry arguments and also the matrix elements of $[L_z, V]$.} The results may be cast in the form $\Delta E = -\vec{d}\cdot\vec{\varepsilon}$. You should verify that the effective dipole moment d turns out to be $ca_0 q$ where a_0 = Bohr radius and c is a dimensionless number of order unity.

$$\varphi_{n\ell m}(\vec{r}) = R_{nl}(r)\, Y_\ell^m(\Omega), \qquad Y_0^0(\Omega) = \frac{1}{\sqrt{4\pi}}, \qquad Y_1^0(\Omega) = \sqrt{\frac{3}{4\pi}}\cos\theta\,,$$

$$\rho = r/a_0, \qquad Y_1^1(\Omega) = -\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{i\Phi} = -Y_1^{-1}(\Omega)^*,$$

$$R_{10}(r) = \frac{4\sqrt{2}\, e^{-\rho/2}}{(2a_0)^{3/2}}, \qquad R_{20}(r) = \frac{(2-\rho)\, e^{-\rho/2}}{(2a_0)^{3/2}}, \qquad R_{21}(r) = \frac{\rho\, e^{-\rho/2}}{\sqrt{3}(2a_0)^{3/2}}\,,$$

$$\int_0^\infty dx\, x^4(2-x)e^{-x} = -72, \qquad \int_0^\infty dx\, x^3 e^{-x} = 6.$$