Candidacy Exam — Spring 2004

QUANTUM MECHANICS

Total points: 100

** Attempt all the problems. There are several parts to each problem with unequal weights, shown in parentheses.

1. [20 points]

This set is lengthy, but only short answers are required.
Consider the eigenvalue equation

$$H|n>= E_n|n>$$

with H the Hamiltonian, $|n>$ the discrete normalized energy eigenstates and E_n the eigenvalues. The time evolution operator is defined as

$$U(t) = e^{-iHt/\hbar} \equiv \sum_n (-iHt/\hbar)^n/n!$$

[2] (a) Evaluate $e^{-iHt/\hbar}|n>$ in terms of $|n>$ and E_n

[2] (b) Evaluate $< n'|H|n>$ in terms of E_n

[2] (c) Evaluate $< n'|e^{-iHt/\hbar}|n>$

[4] (d) Let $|t>$ be the state vector that describes a system at time t. Suppose

$$|t+t_o>= e^{-iHt/\hbar}|t_o>\equiv U(t)|t_o>,$$

and show that therefore $|t>$ satisfies the time-dependent Schrodinger equation.

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(1. ... Continued from previous page.)

[3] (e) Show that $U(t)$ unitary ?

[2] (f) Show that the normalization of $|t>$ is time-independent.

[5] (g) In the position representation, the wavefunction of a single particle is

$$\psi(x,t) = < x|t >= \sum_n c_n \phi_n(x) e^{-iE_n t/\hbar}$$

Write down expressions for the constants c_n and $\phi_n(x)$ in terms of $|n>$, $|x>$ and $|t=0>$.

2. [25 points]

A physicist submits a grant proposal requesting funds to measure the wavelength of visible photons in the following novel way: Construct an *infinite* potential well of width $L = 1$ mm (infinite for all practical purposes); then inject some electrons into the well and measure the wavelength of photons emitted by transfer of electrons between *adjacent* levels. The two situations to be considered are:

(a) "low-n" transitions,

(b) "high-n" transitions.

Evaluate whether the desired information is accesible in each situation. Show your analysis; orders of magnitude estimates are sufficient. Neglect interactions between the electrons. You may use results for the quantum square well without derivation.

Useful Information:
$\hbar = 0.6 \times 10^{-15}$ eV-sec
$m_e = 0.511$ MeV$/c^2$
$c = 3 \times 10^8$ m/sec

3. [30 points]

The Hamiltonian for a system of two *non-identical* (no Pauli principle) spin-1/2 particles is given by

$$H = A(\sigma_{1z} + \sigma_{2z}) + B\vec{\sigma}_1 \cdot \vec{\sigma}_2$$

where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli matrices for particles 1 and 2.

[10 p] (a) Show that the *total* spin operators S_z and $\vec{S}^2$ commute with the Hamiltonian.

[20 p] (b) By choosing an appropriate basis, solve and classify the *eigenvalues* of the Hamiltonian.

4. [25 points]

If the proton in a hydrogen atom is replaced by a positron, one obtains a *positronium* atom. Note that the positron has the same mass as the electron, but opposite charge and spin magnetic moment. The wavefunction of the positronium atom at time $t = 0$ is given by the following superposition of energy eigenfunctions $\psi_{nlm}(\mathbf{r})$:

$$\Psi(\mathbf{r}, t = 0) = \frac{1}{\sqrt{14}}(2\psi_{100}(\mathbf{r}) - 3\psi_{200}(\mathbf{r}) + \psi_{322}(\mathbf{r}))$$

[5 p] (a) Is this wavefunction an eigenfunction of *parity* ? Give reasons.

[8 p] (b) What is the probability of finding the system

(i) in the state (100) ?

(ii) in the state (200) ?

(iii) in the state (322) ?

(iv) in another energy eigenstate ?

[12 p] (c) In the state $\Psi(\mathbf{r}, t = 0)$, what is the expectation value of

(i) the energy ?

(ii) $\mathbf{L}^2$?

(iii) L_z ?