Quantum Mechanics

1) Consider the Zeeman effect (i.e., due to a uniform, time-independent magnetic field) on the ground state of the hydrogen atom and ignore the spin and motion of the proton.

a) Explain why the field-dependent term of the Hamiltonian can be represented by a 2x2 matrix. Give the explicit expression for this term of the Hamiltonian by choosing a quantization direction (or basis) that makes it diagonal.

b) In that basis, consider the state $|\phi> |\chi>$ where

$$|\chi> = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta} \\ e^{i\theta} \end{pmatrix}$$

is due to the magnetic field, the position space ket $|\phi>$ is the properly normalized ground state eigenvector when the magnetic field is turned off, and $\theta$ is real. Calculate the expectation value of the total Hamiltonian in this state.

c) Find a time-dependent expression for $\theta(t)$ that makes $|\phi(t)> |\chi(t)>$ a solution of the equation of motion for this system in the presence of the magnetic field. [Here $|\phi(t)>$ is the time-dependent stationary state corresponding to $|\phi>$.]

2. What are the probabilities that measurements of $L_z$ on the state

$$\psi(\vec{r}) = N(x + y + 2z)e^{-\alpha r}$$

will yield the various allowed values? [Here $\vec{r} = x \hat{x} + y \hat{y} + z \hat{z}$]

[Useful information is:

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta,$$

$$Y_1^{\pm 1}(\theta, \phi) = \pm \sqrt{\frac{3}{8\pi}} e^{\pm i\phi \sin \theta}, \quad Y_2^0(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1).$$]
The one-dimensional oscillator Hamiltonian $\frac{p_x^2}{2m} + \frac{1}{2} m \omega_x^2 x^2$ can be written as $\hbar \omega_x (a_x^\dagger a_x + \frac{1}{2})$ in terms of an operator $a_x$ which is a suitable linear combination of $x$ and $p_x$ satisfying $[a_x, a_x^\dagger] = 1$. Use your knowledge of the properties and solutions of that system (without proof) to consider the two-dimensional system described by

$$
H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2).
$$

a) When $\omega_y = \omega_x = \omega$, what are the energies of the states $a_x^\dagger (a_x^\dagger)^n |\phi_o\rangle$ and $(a_x^\dagger)^m a_x^\dagger |\phi_o\rangle$ where $|\phi_o\rangle$ is the ground state?

b) Determine whether or not the above two states are orthogonal and normalized. Show your work.

c) When $\omega_y = 2\omega_x = 2\omega$, what is the energy and degree of degeneracy of the 3rd excited state? Show your reasoning.

d) In terms of $a_x, a_x^\dagger, a_y, a_y^\dagger$ operators, give an orthogonal basis for the subspace corresponding to the 3rd excited state in c). Show how you know the basis you give is orthogonal.