Quantum Mechanics

1. A particle of mass m scatters from an infinite spherical potential barrier of radius a $\left(V(r) = \infty, r \leq a\right)$. Choose the incident particle to be a plane wave $\psi(r) = e^{ikz}$ with energy $E = \hbar^2 k^2/2m$.

(5 pts.) (a). In the low energy limit, the total cross section is $\sigma \approx 4\pi a^2$. What do you expect (and why) for the high energy limit?

(5 pts.) (b). For what range of energy should the low energy limit be accurate? Give a rough estimate that will likely be valid for deviations less than about 10%.

(20 pts.) (c). Without actually carrying out all the steps necessary for a complete mathematical solution, describe the procedure you would employ to solve this for the scattered wave function and differential cross section. (See **HINTS** on last page.)

2. A particle of mass m is confined in one dimension to the region $0 < x < L$ by infinite potential barriers. The particle is initially $(t < 0)$ in its ground state. At $t = 0$, the barrier at $x = L$ is suddenly removed and the wave function evolves in the region $0 \leq x < \infty$.

(10 pts.) (a). Find the eigenstates of the new Hamiltonian.

(20 pts.) (b). Express the time evolution of the wavefunction as an expansion in the basis of the new $(t > 0)$ Hamiltonian.

(20 pts.) 3. It is often stated that any purely attractive potential in one-dimension has a bound state. For the specific potential $V(x) = -V_o\,\delta(x)$, apply the variational method to the trial wave function

$$\psi_T(x) = \frac{e^{-\left(\frac{x}{2\sigma}\right)^2}}{\left(2\pi\sigma^2\right)^{1/4}}$$

to show that for some choice of σ, the variational energy is negative. Determine the minimum variation energy for this trial function.

4. Three noninteracting particles of mass m are confined to a one-dimensional box, $0 < x < L$. The single particle eigenstates may be labeled

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)\sqrt{\frac{2}{L}}$$

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2 m L^2}.$$

(5 pts.) (a). If these particles are spin-half fermions, two with spin up, $|\uparrow\rangle$, and one with spin down, $|\downarrow\rangle$, what is the totally anti-symmetric ground state?

(5 pts.) (b). For the same three fermions, what is the wave function for the first excited state?

(5 pts.) (c). If these particles were spinless bosons, what would be the totally symmetric first excited state?

(5 pts.) (d). If these three particles were of three different masses (and hence distinguishable) what would be the ground state?

Some Functions, Integrals and Expansions

- Gaussians

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}} dx = 1 .$$

$$\int_{-\infty}^{\infty} x^2 \frac{e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}} dx = \sigma^2 .$$

- Spherical Bessel, Spherical Neuman, Shperical Hankel

$$j_0(x) = \frac{\sin x}{x} \quad (1)$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \quad (2)$$

$$n_0(x) = -\frac{\cos x}{x} \quad (3)$$

$$n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \quad (4)$$

$$h_0^{(1)}(x) = \frac{-i\, e^{ix}}{x} \quad (5)$$

$$h_0^{(2)}(x) = \frac{i\, e^{-ix}}{x} \quad (6)$$

$$h_n^{(1)}(x) = j_n(x) + i\, n_n(x) \quad (7)$$

$$h_n^{(2)}(x) = j_n(x) - i\, n_n(x) \quad (8)$$

(9)

- Plane wave

$$e^{ikz} = \sum_{n=0}^{\infty}(2l+1)i^l j_l(kr) P_l(\cos\theta)$$