(1) (30 pt.) Consider a very small mirror suspended from a quartz strand with elastic constant D (i.e., the torque on the strand is given by $-D\theta$, or the energy in the strand by $\frac{1}{2}D\theta^2$), in equilibrium with a gas. The mirror reflects a laser beam in such a way that the angular fluctuations (Brownian motion) caused by the impact of surrounding molecules can be measured. The equilibrium position of the mirror is $\langle \theta \rangle = 0$. At $T=287$ K, the mean squared angular displacement was measured to be $\langle \theta^2 \rangle = 4.20 \cdot 10^{-6}$ for a strand with $D = 9.43 \cdot 10^{-16} \text{ N} \cdot \text{m}$.

(a; 25 pt) Calculate Avogadro's number from these experimental results. You may use the universal gas constant $R = 8.31 \text{ J} \cdot \text{K}^{-1} \cdot \text{mol}^{-1}$.

(b; 5 pt) Can the amplitude of these fluctuations be reduced by reducing the gas density? Explain why or why not.

(2) (40 pts) Consider a very crude model of a rubber band: a chain of $N$ links, each of length $\ell$ (see figure). Each link has only two possible states, pointing in either the $+x$ or $-x$ direction; the states have equal energy, which means equal probability. The total length $L$ of the rubber band is the net displacement from the beginning of the first link to the end of the last link.

(a; 15 pts) Find an expression for the entropy of this system in terms of $N$ and $L$. It is acceptable to leave undetermined a constant that depends on $N$ only. Hint: there are at least two acceptable approaches here:

(i) Consider the relationship between $L$ and the total number of steps in the forward direction. If you choose this method, you may want to use the Stirling approximation, $\ln N! \approx N \ln N - N$, as well as the approximation

$$\ln(1+x) \approx x - \frac{1}{2} x^2 + ...$$

(ii) Consider the probability distribution for $L$.

(b; 10 pts) For a one-dimensional system such as this, the length $L$ is analogous to the volume $V$ of a three-dimensional system, and the force $f$ on the band is analogous to the pressure. Take $f=0$ for the band pulling in. From this analogy, write down an expression for the internal energy $U(L,S)$ in terms of $f$, $L$, $T$, and $S$.

(c; 10 pts) Find the free energy $F(L,T)$ as the appropriate Legendre transformation of $U(L,S)$.

(d; 10 pts) Express the tension force $F$ in terms of a partial derivative of the entropy. From this expression, compute the tension in terms of $L$, $T$, $N$, and $\ell$.

(3) (30 pts) Consider Bose condensation for system in $D$ dimensional space with the relation between energy and momentum given by $\varepsilon \propto |p^D|$. Find a relation between $D$ and $\sigma$ for Bose condensation to occur.