Candidacy Exam Fall 2009

Statistical Mechanics

[30 pts] 1. Consider an array of N non-interacting atoms that obey Maxwell-Boltmann statistics. Each atom has three non-degenerate states with energies: $\epsilon_1 < \epsilon_2 < \epsilon_3$ and the system is at equilibrium with temperature T.

(a) What is the probability for all the atoms to be in the ground state?

(b) What is the probability that a third of the atoms is in the ϵ_1 state, a third of the atoms is in the ϵ_2 state, and a third of the atoms is in the ϵ_3 state. (Assume that N is a multiple of 3)?

(c) Describe the state of the system for very low temperature ($k_B T \ll \epsilon_1$) as well as very high temperature ($k_B T \gg \epsilon_3$). Write down expressions for the free energy per atom for each limit.

[35 pts] 2. Spin waves, produced by fluctuations in the transverse components of spins, are low energy excitations above the ferromagnetic state. The dispersion relation for a spin wave with frequency ω and wavevector $\mathbf{k}$ is given by

$$\omega = A|\mathbf{k}|^2. \tag{1}$$

where A depends of the interactions between the spins. Quantized spin waves, magnons, are non-interacting bosons whose number is not conserved.

(a) Show that the density of states for magnons can be written as

$$\mathcal{D}(\epsilon)\mathrm{d}\epsilon = C\sqrt{\epsilon}\mathrm{d}\epsilon \tag{2}$$

and find C in terms of Planck's constant, A, and the volume V.

(b) Calculate the number of magnons N at equilibrium at temperature T.

Express your answers in terms of the dimensionless integrals

$$I_1 = \int_0^\infty \frac{\sqrt{x}\mathrm{d}x}{e^x - 1} \qquad I_2 = \int_0^\infty \frac{x^{3/2}\mathrm{d}x}{e^x - 1} \tag{3}$$

Problem 3 is given on the next page.

[35 pts] 3. Consider a dilute gas of N non-interacting molecules with dipole moment $\mathbf{p}$ in an external field $\mathbf{E} = E\hat{\mathbf{z}}$. The classical Hamiltonian describing rotational degrees of freedom of a molecule can be written in terms of the angular velocities

$$\mathcal{H}_{\text{rot}}(\dot{\theta}, \dot{\varphi}, \theta, \varphi) = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}I\sin^2\theta\dot{\varphi}^2 - Ep\cos\theta \tag{4}$$

where I is the moment of inertia, θ is the angle between $\mathbf{p}$ and $\mathbf{E}$, and φ is the corresponding azimuthal angle.

Neglecting vibrations, the partition function can be written as

$$Z = \frac{Z_1^N}{N!} \qquad \text{with} \qquad Z_1 = Z_{\text{trans}} Z_{\text{rot}} \tag{5}$$

where Z_{trans} and Z_{rot} are the partition functions corresponding the translations and rotations of a single molecule.

(a) Find the canonical momenta p_θ and p_φ and write the Hamiltonian $\mathcal{H}_{\text{rot}}(p_\theta, p_\varphi, \theta, \varphi)$.

(b) Show that

$$Z_{\text{rot}} = A\frac{\sinh\beta pE}{\beta pE} \tag{6}$$

and find A in terms of I, $\hbar$, and $\beta = 1/k_{\text{B}}T$.

(c) Calculate the polarization of the gas

$$P = \frac{N}{V}\langle p\cos\theta\rangle \tag{7}$$

You may find the following relation useful:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} = \sqrt{\frac{\pi}{a}} \tag{8}$$