Thermodynamics and Statistical Mechanics

Instructions: Solve all three problems.

1. (30 points) The molar specific heat at constant volume of a monatomic ideal gas is known to be $3R/2$. Suppose that one mole of such a gas undergoes a cyclic quasistatic process which appears as a circle in the $P-V$ plane shown below. ($P_0$ and $V_0$ are a reference pressure and volume, respectively).

   (a) Calculate the net work done by the gas in one cycle in terms of the product $P_0V_0$. Hint: Consider $P dV = P_0V_0 \frac{P}{P_0} d \left( \frac{V}{V_0} \right)$.

   (b) Calculate the internal energy change of the gas between state $C$ and state $A$ in terms of $P_0V_0$.

   (c) Calculate the heat absorbed by the gas in going from $A$ to $C$ via the path $ABC$ of the cycle, in terms of $P_0V_0$.

2. (40 points) Consider a system of $N$ localized particles with non-degenerate energy levels given by $\varepsilon_0 = 0$, $\varepsilon_1 = \Delta$.

   (a) Calculate the canonical partition function $Z_C$.

   (b) Calculate the average energy $\overline{E}$ of the system in contact with a heat reservoir.
(c) Given \( \overline{E^2} - \overline{E}^2 = -\frac{\partial}{\partial \beta} \overline{E} \), calculate the fractional rms energy fluctuation,

\[
f = \left( \frac{\overline{E^2} - \overline{E}^2}{\overline{E}^2} \right)^{1/2},
\]

of the system in terms of \( N, \beta, \) and \( \Delta. \)

(d) Suppose \( \Delta = 10^{-16} \text{ erg} \) and there are \( 10^{23} \) particles. Estimate the temperature range for which \( f \geq 1. \) (Recall \( k_B = 1.38 \times 10^{-16} \text{ erg/K.} \))

3. (30 points) Suppose the quantum state of a harmonic crystal consisting of \( N \) atoms is given by a list of \( 3N \) quantum numbers \( \{n_1, n_2, \ldots, n_{3N}\} \), each of which is a non-negative integer \( (n_i = 0,1,2,\ldots) \). The corresponding energy levels of the system are given by \( \varepsilon_i = (n_i + 1/2) \hbar \omega_i. \)

(a) Calculate the mean energy of the system at temperature \( T. \)

(b) Assume the oscillator frequencies do not depend on \( T \). Evaluate the energy in the limit \( T \to 0 \) and explain why it is non-zero.