I–B Part 1: A particle of mass m at a distance R from another mass M is bound by a gravitational potential energy $V(R) = -GMm/R$. If the particle moves at escape velocity $v_{\rm esc}$, it has just enough kinetic energy to reach infinity, *i.e.*,

$$\frac{mv_{\rm esc}^2}{2} = \frac{GMm}{R} \qquad \text{or} \qquad v_{\rm esc} = \sqrt{\frac{2GM}{R}} \tag{1}$$

For a circular orbit under a gravitational force $F_{\rm grav} = GMm/R^2$, the orbital speed v is constant and we can equate $F_{\rm grav}$ to

$$\frac{mv^2}{R} = \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2$$

$$i.e., \quad \frac{GMm}{R^2} = \frac{4\pi^2 mR}{T^2}$$

$$\text{hence} \quad R = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} \tag{2}$$

Part 2: Let F_c be the cable tension at the counterweight. The centripetal force on the counterweight is partly from gravity acting on the counterweight, and the remainder is from the cable tension:

$$\frac{mv_c^2}{R_c} = \frac{GMm}{R_c^2} + F_c$$

where R_c is the radius of the counterweight's orbit. Using $v_c = v_{\rm esc}$ and Eq. (1), we can eliminate v_c and express the tension as:

$$F_c = \frac{m}{R_c}\left(\frac{2GM}{R_c}\right) - \frac{GMm}{R_c^2} = \frac{GMm}{R_c^2}$$

Next, we need an expression for R_c, the radius of the counterweight's orbit, in terms of the given quantities:

$$v_{\rm esc} = v_c = \frac{2\pi R_c}{T}; \qquad \text{therefore} \quad R_c = \frac{T}{2\pi}\sqrt{\frac{2GM}{R_c}}$$

$$\text{and} \quad R_c = \left(\frac{GMT^2}{2\pi^2}\right)^{1/3}$$

Thus we conclude that

$$F_c = m\left(\frac{4\pi^4 GM}{T^4}\right)^{1/3}$$

Note that F_c above gives the cable tension at the counterweight regardless of whether or not the weight of the cable itself needs to be taken into account.

Part 3: The following is an outline of the qualitative argument. The optimum thickness of the beanstalk cable needs to increase or decrease as the total tension in the cable increases or decreases. Consider what happens as we move inward from the counterweight, where the tension is F_c (as calculated in part 2 above). Every part of the cable between the geosynchronous satellite and the counterweight is moving *faster* than the speed of an untethered satellite in a circular orbit at the same radius. (Remember that the counterweight's orbital speed equals escape velocity.) Allowing for the finite mass of the cable itself, the tension and the minimum required cable strength increase as we move inward towards the geosynchronous satellite. The geosynchronous satellite maintains its orbit regardless of the cable. At this orbit, the outward pull of the counterweight and its part of the cable is balanced by the inward force due to the weight of the inner part of the cable plus the tension F_e at the point of attachment to the earth. The inner part of the cable orbits more slowly than an untethered satellite in a circular orbit at the same radius and would fall to earth if the cable were to break. At the surface of the earth, only F_e remains. Therefore, the required cable strength and thickness decrease as we move away in either direction from the geosynchronous satellite.

Part 4: To simulate $g = 9.8\ \mathrm{m/s^2}$ at a radius of 1 km, the space colony needs to rotate at an angular velocity $\omega = \sqrt{(g/R)} = 0.099$ radians/sec. Thus the counterweight radius R_c needed for $v_c = 3$ km/s is $R_c = v_c/\omega = 30$ km.

I–C Part (1): For Allo and his bike to stay on the track at the top of the loop, we equate

$$(T+V)_{\text{start}} = (T+V)_{\text{top of loop}}$$

or

$$mgh' = \tfrac{1}{2}mv_{\text{min}}^2 + 2mgR$$

where v_{min} is the speed at the top of the loop; the minimum needed to stay on the track corresponds to $v_{\text{min}}^2/R = g$. Thus

$$mgh' = \tfrac{1}{2}mgR + 2mgR \quad \text{or} \quad h' = 5R/2$$

Part (2): On the ramp

$$\tfrac{1}{2}mv^2 = mg[h' - h(s)] \tag{3}$$

$$\text{and} \quad v^2 = \dot{s}^2 + \dot{h}^2$$
$$= \dot{s}^2 + \left(\frac{dh}{ds}\,\dot{s}\right)^2 = \dot{s}^2\left[1 + \left(\frac{dh}{ds}\right)^2\right]$$

Reinserting into Eq. (3), we get

$$\tfrac{1}{2}m\dot{s}^2\left[1 + \left(\frac{dh}{ds}\right)^2\right] = mg(h' - h)$$

$$\dot{s} = \left[\frac{2g(h'-h)}{1+(dh/ds)^2}\right]^{1/2}$$

Part (3): We are told to start from the above expression for $\dot{s}$; thus

$$dt = \left[\frac{1+(dh/ds)^2}{2g(h'-h)}\right]^{1/2} ds$$

and the time taken to reach the bottom is

$$\Delta t = \int_{s'}^{0} \left[\frac{1+(dh/ds)^2}{2g(h'-h)}\right]^{1/2} ds$$

where $s = s'$ when $h = h'$. In this problem, we are told that the ramp has a uniform slope, i.e., $h = as$ where a is constant; thus $dh/ds = a$ and $h' = as'$. Consequently,

$$\Delta t = \sqrt{\frac{1+a^2}{2ga}} \int_{s'}^{0} \frac{ds}{\sqrt{s'-s}}$$

The integral term is

$$\int_{s'-s=0}^{s'-s=s'} \frac{-d(s'-s)}{\sqrt{s'-s}} \;=\; -2\sqrt{s'} \;=\; -2\sqrt{h'/a}\,,$$

where the minus sign signifies that s decreases with time. Thus,

$$\Delta t = \sqrt{\frac{2h'(1+a^2)}{ga^2}} = \sqrt{\frac{2h'}{g}\left(1+\frac{1}{a^2}\right)}$$