Sample Solutions — 1998 Classical Mechanics Homework Set II

II–A. (a) Apply eq. (1-70) in Goldstein to a single vertical coordinate z, pointing downward. Eq. (1-70) is basically Lagrange's eq. with an added term to account for the frictional energy, which cannot be be made part of the potential V:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} + \frac{\partial \mathcal{F}}{\partial \dot{z}} = 0$$

Then

$$T = \frac{1}{2}m\dot{z}^2\,, \qquad V = -mgz \quad \text{and} \quad \mathcal{F} = b\dot{z}^2\,.$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}\,, \qquad \frac{\partial L}{\partial z} = mg \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial \dot{z}} = 2b\dot{z}$$

and the equation of motion is

$$m\ddot{z} - mg + 2b\dot{z} = 0 \qquad \text{or} \qquad \frac{dv}{dt} = g - \frac{2b}{m}v,$$

where we have changed notation to $v = \dot{z}$. We can now integrate:

$$\int \frac{dv}{g - 2bv/m} = \int dt \quad \text{and so} \quad \ln(g - \frac{2b}{m}v) = -\frac{2b}{m}t + C$$

Using our initial condition that $v(t=0) = v_0$, we get $C = \ln(g - 2bv_0/m)$ and so

$$\ln\left(\frac{g - 2bv/m}{g - 2bv_0/m}\right) = -\frac{2b}{m}t$$

$$\frac{2bv - mg}{2bv_0 - mg} = \exp(-2bt/m)$$

$$2bv = (2bv_0 - mg)\exp(-2bt/m) + mg$$

$$v = (v_0 - \frac{mg}{2b})\exp(-2bt/m) + \frac{mg}{2b}$$

$$v(t) = \frac{mg}{2b}[1 - \exp(-2bt/m)] + v_0\exp(-2bt/m)$$

(b) In this part, the only difference is that now there is no gravitational force. Therefore, we set all terms with g to zero, to get

$$v(t) = v_0\exp(-2bt/m)$$

II–B. For the given Lagrangian,

$$\frac{\partial L}{\partial \dot{q}} = e^{bt} m\dot{q} \quad \text{and} \quad \frac{\partial L}{\partial q} = e^{bt} kq$$

Thus Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad \Rightarrow \quad e^{bt}(m\ddot{q} + bm\dot{q} + kq) = 0$$

Unless $t \to -\infty$, any solution must satisfy

$$\ddot{q} + b\dot{q} + \frac{k}{m}q = 0\,. \tag{1}$$

Eq. (1) is the standard differential equation for a damped harmonic oscillator, where b represents the strength of damping and k is the spring constant; the special case $b = 0$ is a simple harmonic oscillator.

Knowing the physical characteristics of a damped harmonic oscillator, we can immediately state that there are no constants of the motion. In chapter 2, we will consider a more formal approach to constants of the motion — each instance of a generalized coordinate or time not explicitly appearing in the Lagrangian leads to a constant of the motion. In this instance, $L = L(q, t)$ and so there are no constants of the motion.

The following is an abbreviated outline of the possible solutions to Eq. (1); for full details, see any standard undergraduate mechanics textbook. Begin by testing a solution of the form $q \propto e^{\alpha t}$. Inserting in Eq. (1), we get

$$\alpha^2 + b\alpha + \frac{k}{m} = 0$$

This quadratic has two roots:

$$\alpha = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{k}{m}}.$$

There are three categories of possible motion according to the value of the square root term

$$\beta = \sqrt{\frac{b^2}{4} - \frac{k}{m}}\,.$$

First, an imaginary $\beta = i\omega$, where ω is real, corresponds to decaying oscillatory motion (*"underdamped"*) of the form

$$q(t) = q_0 e^{-bt/2} e^{i\omega t + \phi}\,.$$

Second, $\beta = 0$ corresponds to

$$q(t) = q_0 e^{-bt/2}\,,$$

known as *critically damped* motion. Third, if β is real and non-zero, then

$$q(t) = e^{-bt/2}(q_0 e^{\beta t} + q_0' e^{-\beta t})\,,$$

and the motion is said to be *overdamped.* In both critically damped and overdamped cases, the motion is not oscillatory; if there is an initial displacement q_0, the system returns to the equilibrium position $q = 0$, and does so in the shortest possible time when the motion is critically damped.

II–C. For this spring pendulum, let $r(t)$ be the distance of the mass from the pivot point of the spring, and let $\theta(t)$ be the angle of the spring from the vertical. These two generalized coordinates are independent of each other, and fully specify the position of the mass, given the constraint that the motion is confined to a vertical plane.

The velocity component associated with the r coordinate is $\dot{r}$ and the velocity component associated with the θ coordinate is $r\dot{\theta}$. Hence the kinetic energy is

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)\,.$$

The potential energy has an extra term compared to the case of the spherical pendulum worked earlier — now we must include the compressional energy of the spring:

$$V = \tfrac{1}{2}k(r-\ell)^2 - mgr\cos\theta\,.$$

Inserting $L = T - V$ above into Lagrange's eqs. for r and θ, we get the two equations of motion (EOM):

$$m\ddot{r} - mr\dot{\theta}^2 + k(r-\ell) - mg\cos\theta = 0$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0\,.$$

In our notation, the equilibrium coordinates of the spring are

$$(r_0,\ \theta_0) = (\ell + \frac{mg}{k},\ 0)\,.$$

Let $s = r - r_0$. For small oscillations, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. In this notation, the EOM become

$$m\ddot{s} - m(r_0 + s)\dot{\theta}^2 + ks = 0 \qquad \text{and} \qquad (r_0 + s)\ddot{\theta} + 2\dot{s}\dot{\theta} + g\theta = 0$$

This still leaves us with a relatively complicated pair of equations, each including terms in both coordinates. However, the small oscillation approximation allows further simplification, since $s \ll r_0$, and we can also neglect terms beyond the lowest order in the small quantities $\dot{s}$ and $\dot{\theta}$. Thus we end up with two independent simple harmonic oscillator equations:

$$\ddot{s} + \frac{k}{m} s = 0 \qquad \text{and} \qquad \ddot{\theta} + \frac{g}{r_0}\theta = 0$$

These have the usual well-known solutions; they are written below in terms of the original coordinates r and θ and parameters m, ℓ and k:

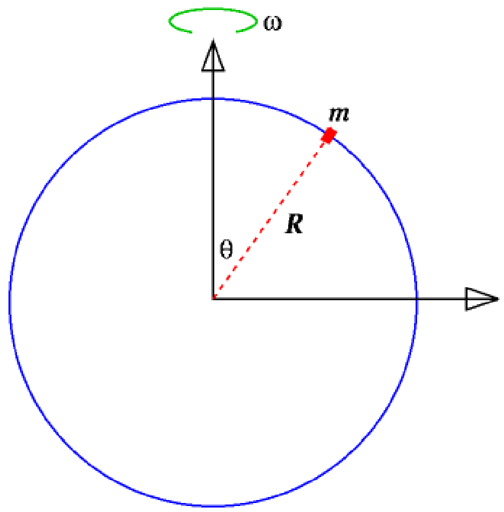
$$r = \ell + \frac{mg}{k} + R_0 \cos\left(\sqrt{\frac{k}{m}}\, t + \phi\right) \,,$$

and

$$\theta = \Theta_0 \cos\left(\sqrt{\frac{kg}{kl + mg}}\, t + \phi'\right) \,,$$

where the amplitudes R_0 and Θ_0 and the phase angles ϕ and ϕ' are constants of integration and are fixed by the initial conditions.

II–D. In this problem, it is OK to take our coordinate system to be fixed relative to the rotating hoop as long as we allow for the centrifugal force seen in the rotating system. Because the hoop constrains the bead to move on the circumference of a circle of fixed radius R, we require only one generalized coordinate — the angle θ of the bead relative to the center of the hoop, as in the diagram below:



If the bead is at a fixed angle θ, the rotation of the hoop makes it trace out a circle of radius $r = R\sin\theta$. The contribution to the potential coming from the centrifugal force is given by

$$-\frac{\partial V_{\text{cent}}}{\partial r} = m\omega^2 r \,,$$

$$V_{\text{cent}} = -\tfrac{1}{2}m\omega^2 r^2 = -\tfrac{1}{2}m\omega^2 R^2 \sin^2\theta\,.$$

The total potential is $V = V_{\text{cent}} + V_{\text{grav}}$, where the gravitational potential, relative to the center of the hoop, is $V_{\text{grav}} = mgR\cos\theta$. The kinetic energy is $T = mR^2\dot{\theta}^2/2$, so the Lagrangian $T - V$ is

$$L = \tfrac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgR\cos\theta\,.$$

Lagrange's equation for θ thus gives the EOM

$$\ddot{\theta} - \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta = 0$$

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Consider a case where V is dominated by gravity (due to ω being small), and where the bead is released at rest near (but not exactly at) the bottom of the hoop. The bead will oscillate about $\theta = 180°$, with gravity supplying the restoring force. However, if ω is very large so that V is dominated by the centrifugal term, a bead released under the same conditions will "flee the center" and its equilibrium angle θ will be just over $90°$ or just under $270°$ (depending on which is closer to the initial release point). In order to quantify the condition on ω for the first scenario to happen, we define δ as a small deflection angle relative to the bottom of the hoop, i.e., $\delta = \theta - 180°$. Then

$$\sin\theta = \sin(180° + \delta) = -\sin\delta \approx -\delta\,,$$

$$\cos\theta = \cos(180° + \delta) = -\cos\delta \approx -1\,.$$

Now the EOM becomes

$$\ddot{\delta} + \left(\frac{g}{R} - \omega^2\right)\delta = 0\,.$$

This is the EOM for simple harmonic motion about an equilibrium point at $\delta = 0$ only as long as the term in parentheses is *positive*, i.e., we require

$$\omega < \sqrt{(g/R)}\,.$$