Sample Solutions — 1998 Classical Mechanics Homework Set III

III–A. Choose cylindrical coordinates (ρ, φ, z), with the z axis vertical (as specified in the question) and with the origin at the lowest point, at the center of the mirror. In cylindrical coordinates, the velocity has two linear components $\dot\rho$ and $\dot z$, and another component $\rho\dot\varphi$ along a circular arc centered on the z axis. Thus the Lagrangian is

$$L = T - V = \tfrac{1}{2}m(\dot\rho^2 + \rho^2\dot\varphi^2 + \dot z^2) - mgz\,.$$

The constraint equation in cylindrical coords is

$$f(\rho, \varphi, z) = \rho^2 - az = 0$$

Our alternative form of Lagrange's eqs. (eq. [16] in lecture notes) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} = Q_k\,, \tag{1}$$

where we now use slightly different notation to write the generalized forces of constraint Q_k in terms of Lagrange's Multipliers λ_l (one for each constraint eq. $f_l = 0$) using

$$Q_k = \sum_l \lambda_l \frac{\partial f_l}{\partial q_k}\,. \tag{2}$$

In the present problem, we have just one constraint eq. and one multiplier λ. Therefore, the above equation for Q_k gives us

$$Q_\rho = 2\rho\lambda\,, \qquad Q_\varphi = 0 \quad \text{and} \quad Q_z = -a\lambda\,, \tag{3}$$

while the corresponding Lagrange's eqs. for ρ, φ and z are

$$m\ddot\rho - m\rho\dot\varphi^2 = 2\rho\lambda\,, \tag{4}$$

$$\frac{d}{dt}(m\rho^2\dot\varphi) = 0\,, \tag{5}$$

$$m\ddot z + mg = -a\lambda\,. \tag{6}$$

To find the constraint force, we must express λ in terms of the given quantities, namely ρ only, and the given constants. Start with eq. (6), since it contains λ with just one unknown, i.e., $\ddot z$. Use the constraint eq. to substitute for the latter:

$$az = \rho^2 \quad \Rightarrow \quad a\dot z = 2\rho\dot\rho \quad \Rightarrow \quad a\ddot z = 2\rho\ddot\rho + 2\dot\rho^2 \tag{7}$$

Thus, eq. (6) and the above $\Rightarrow$

$$\frac{2m\rho\ddot\rho}{a} + \frac{2m\dot\rho^2}{a} + mg + a\lambda = 0\,.$$

Using eq. (4) we get rid of $\ddot{\rho}$:

$$\frac{2\rho}{a}(2\rho\lambda + m\rho\dot{\varphi}^2) + \frac{2m\dot{\rho}^2}{a} + mg + a\lambda = 0\,.$$

Then, eq. (5) $\Rightarrow$ $m\rho^2\dot{\varphi} = \text{const} = J$, so

$$\frac{4\rho^2\lambda}{a} + \frac{2J^2}{ma\rho^2} + \frac{2m\dot{\rho}^2}{a} + mg + a\lambda = 0\,,$$

$$a\lambda\left(1 + \frac{4\rho^2}{a^2}\right) + \frac{2J^2}{ma\rho^2} + \frac{2m\dot{\rho}^2}{a} + mg = 0\,. \tag{8}$$

This leaves us with only $\dot{\rho}$ to eliminate. Note also that we are given the total energy E, which can be written

$$E = T + V = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) + mgz\,.$$

Now substitute for z and $\dot{z}$ using eq. (7), and for $\dot{\varphi}$ using eq. (5):

$$\frac{2E}{m} = \dot{\rho}^2 + \frac{J^2}{m^2\rho^2} + \frac{4\rho^2\dot{\rho}^2}{a^2} + 2g\frac{\rho^2}{a}\,.$$

Now collect terms in $\dot{\rho}$ on the left:

$$\dot{\rho}^2\left(1 + \frac{4\rho^2}{a^2}\right) = \frac{2E}{m} - \frac{J^2}{m^2\rho^2} - 2g\frac{\rho^2}{a}\,.$$

Combining the above with eq. (8), we get an expression for λ using only the given quantities:

$$a\lambda = -\left(\frac{4E}{a} + \frac{8J^2}{ma^3} + mg\right)\left(1 + \frac{4\rho^2}{a^2}\right)^{-2}$$

Finally, eq. (3) tells us that the three components of the force of constraint $\mathbf{f}$ are

$$\mathbf{f} = (2\rho\lambda,\;\; 0,\; -a\lambda)\,,$$

$$|\mathbf{f}| = \left(\frac{4E}{a} + \frac{8J^2}{ma^3} + mg\right)\left(1 + \frac{4\rho^2}{a^2}\right)^{-3/2}\,.$$

III–B. (1) The problem of a particle sliding without friction in a spherical bowl is identical to the problem of the spherical pendulum worked earlier. Therefore, the Lagrangian is

$$L = \frac{mR^2}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 + \frac{2g}{R}\cos\theta)$$

(2)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \qquad \text{and} \qquad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mR^2\dot{\varphi}\sin^2\theta\,.$$

(3) The Lagrangian is cyclic in time and in φ. Because it is cyclic in time, the Hamiltonian is conserved. In this conservative system, that just means that the mechanical energy $T + V$ is conserved.

Because the Lagrangian is cyclic in φ, the conjugate momentum p_φ is conserved. As discussed in the context of the spherical pendulum, $mR^2\dot{\varphi}\sin^2\theta$ is the angular momentum about the vertical axis. This is the only component of angular momentum that is conserved, because in the presence of gravity and the constraint of the bowl, rotational symmetry exists only for rotations about the vertical axis.

(4) Refer back to special case #2 under 1997 Homework II-B solutions. For constant polar angle θ_0, we find

$$\dot{\varphi} = \pm\sqrt{g\,/R\cos\theta_0}\,,$$

so the velocity $v_\varphi = R\sin\theta_0\,\dot{\varphi}$ is

$$v_\varphi = \pm\sqrt{gR\sin^2\theta_0/\cos\theta_0}\,.$$

The $\pm$ sign signifies that uniform angular velocity in either direction along a circular path satisfies the initial condition $\theta = \theta_0$.

III–C. We are given $q = s\exp(-bt/2)$; then

$$\dot{q} = (\dot{s} - \tfrac{1}{2}bs)\exp(-\tfrac{1}{2}bt)$$

and the Lagrangian becomes

$$L = \tfrac{1}{2}m(\dot{s} - \tfrac{1}{2}bs)^2 - \tfrac{1}{2}ks^2\,. \tag{9}$$

Lagrange's eq. for our new variable s is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \qquad \Rightarrow \qquad \ddot{s} + \left(\frac{k}{m} - \frac{b^2}{4}\right)s = 0$$

This looks exactly like the EOM for a simple harmonic oscillator with angular frequency given by $\omega^2 = k/m - b^2/4$. The well-known constant of the motion for a simple harmonic oscillator is the mechanical energy $T + V$ which is proportional to $\dot{s}^2 + \omega^2 s^2$. Furthermore, the Lagrangian given by eq. (9) above no longer depends explicitly on the time, which formally points to a conserved energy. However, all of the above statements are misleading, because the time dependence is always hiding inside the "coordinate" s. In fact, s is not really a generalized coordinate in our normal sense of that term. When the presence of the $\exp(bt/2)$ term in s is taken into account, it becomes clear that the motion must be exactly as described in the solution to the first part (see homework II-B).