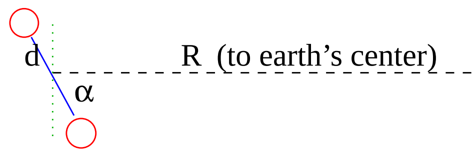
Sample Solutions — 1998 Classical Mechanics Homework Set IV

IV–A. This problem involves only manipulation of the Newtonian gravitational potential $-GMm/R$. Suppose that the satellite is oriented so that the line joining the center of the two spheres, a distance $2d$, makes an angle α with the line to the center of the body which is being orbited. It is important to define the angle α such that the center of mass of the satellite does not move when α is varied:



$$V = -GMm\left[\frac{1}{R + d\cos\alpha} + \frac{1}{R - d\cos\alpha}\right].$$

$$\frac{\partial V}{\partial \alpha} = GMm\left[\frac{(-d\sin\alpha)}{(R + d\cos\alpha)^2} + \frac{d\sin\alpha}{(R - d\cos\alpha)^2}\right].$$

It can be seen that either $\sin\alpha = 0$ **or** $\cos\alpha = 0$ results in $\partial V/\partial\alpha = 0$. Checking the second derivatives, we find that the orientation $\alpha = 0$ corresponds to a minimum in V, while the orientation $\alpha = 90°$ maximizes V. Therefore, we conclude that gravity tends to automatically stabilize the satellite with its long axis pointing towards the center of the body being orbited.

IV–B. We follow the standard procedure to make this a one-dimensional problem, i.e., add the centrifugal term to the central force or potential. Thus

$$f' = f + \frac{\ell^2}{mr^3} = -\frac{d^2\ell^2}{mr^5}. \qquad (1)$$

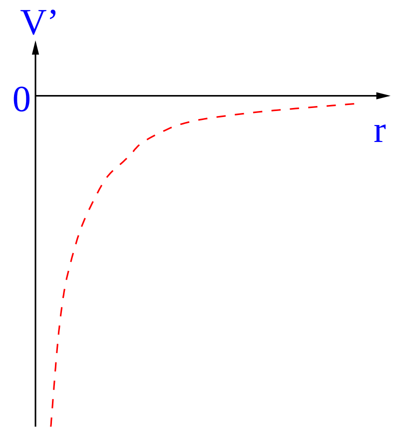
The corresponding effective one-dimensional potential is

$$V'(r) = -\frac{d^2\ell^2}{4mr^4}.$$

We have taken $V'(\infty) = 0$, as usual. The sketch illustrates the qualitative form of this $1/r^4$ potential.

The given spiral path with uniform pitch c determines what the form of the force must be; to find that form, we just need to take the 2nd time derivative:

$$\dot r = c\dot\theta = c\frac{\ell}{mr^2} \quad \text{and} \quad \ddot r = -\frac{2c\ell}{mr^3}\dot r = -\frac{2c^2\ell^2}{m^2r^5}.$$



Thus, $m\ddot{r} = f' = -2c^2\ell^2/mr^5$ is the same form as eq. (1) above, proving that a spiral path $r = r_0 + (d/\sqrt{2})\theta$ leads to the given force.

The total energy is $E = T_r + T_\theta + V = m\dot{r}^2/2 + V'$, and for a particle in a spiral with uniform pitch, we have seen that

$$\frac{m\dot{r}^2}{2} = \frac{m}{2}\left(\frac{c\ell}{mr^2}\right)^2.$$

Thus we are led to the conclusion that $E = 0$ if $c^2 = d^2/2$, whereas if $c^2 \neq d^2/2$, we have already shown that a spiral of uniform pitch is *not* a solution. Summarizing this, plus what we can tell from the form of the potential V', we conclude

If $E = 0$, **motion is unbounded** [and path is spiral $r = r_0 + (d/\sqrt{2})\theta$].
If $E > 0$, **motion is unbounded**.
If $E < 0$, **motion is bounded**.

IV–C. Once again, the given path determines what form of the force or potential must have; to find that form, we just need to differentiate:

$$r = c\theta^2 \quad \Rightarrow \quad \dot{r} = 2c\theta\dot{\theta} = 2c\sqrt{\frac{r}{c}}\,\frac{\ell}{mr^2} = \frac{2\ell\sqrt{c}}{m}r^{-3/2}.$$

Using total energy $E = T_r + T_\theta + V \;=\; m\dot{r}^2/2 + \ell^2/2mr^2 + V$, we have

$$V = E - \frac{m\dot{r}^2}{2} - \frac{\ell^2}{2mr^2}$$

$$\propto E + Ar^{-3} + Br^{-2}.$$

This agrees with the given form of the potential if we choose our zero reference position for potential energy such that $E = 0$, and therefore, one of the exponents n or m must be -3 and the other must be -2.

IV–D. (1) Let the planet be a distance r from the center of the star. We apply Gauss' law to find the gravitational field at r due to the dust cloud. Thus, the attractive gravitational force $f = -GmM_{\rm tot}/r^2$ consists of two terms:

$$f(r) \approx -\frac{Gm}{r^2}\,(M + \tfrac{4}{3}\pi r^3 \rho)$$

Since we are not given the radius of the star (let's call it R_s), this approximation works best when $R_s \ll r$ and/or when $\rho \ll M/R_s^3$.

(2) Our standard expression for the one-dimensional treatment of central force problems is

$$m\ddot{r} = f'(r) = f(r) + \frac{\ell^2}{mr^3}\,.$$

For a circular orbit at radius $r = R$, $\ddot{r} = 0$ and so we obtain the desired polynomial expression

$$-\frac{Gm}{R^2}\,(M + \tfrac{4}{3}\pi R^3 \rho) + \frac{\ell^2}{mR^3} = 0\,. \tag{2}$$

(3) The left-hand side of eq. (2) is the effective one-dimensional force $f'(R) = 0$ on the planet in the circular orbit. The corresponding effective potential is related to this force by $f'(R) = -(\partial V'/\partial r)_{r=R} = 0$. We know that there is a minimum in V' at $r = R$, so we can define a new variable $s = r - R$ and expand in a Taylor series in s:

$$V'(s) \approx V'(s=0) + [\text{zero linear term}] + \tfrac{1}{2}s^2 \left(\frac{d^2V'}{dr^2}\right)_{r=R}\,.$$

Next we write the total energy E (a constant) in terms of

$$E = T_r + V' = \tfrac{1}{2}m\dot{s}^2 + \tfrac{1}{2}s^2 \left(\frac{d^2V'}{dr^2}\right)_{r=R}\,.$$

Now differentiate with respect to time:

$$0 = m\ddot{s} + s\left(\frac{d^2V'}{dr^2}\right)_{r=R}\,.$$

Thus we find that small deviations are described by simple harmonic motion in s with angular frequency

$$\omega_s = \sqrt{\frac{1}{m}\left(\frac{d^2V'}{dr^2}\right)_{r=R}}\,. \tag{3}$$

Now differentiate f', i.e. the left-hand side of eq. (2), then use eq. (2) again to eliminate the first term:

$$\left(\frac{d^2V'}{dr^2}\right)_{r=R} = -\frac{2GMm}{R^3} + \tfrac{4}{3}\pi\rho Gm + \frac{3\ell^2}{mR^4} \;\approx\; 4\pi\rho Gm + \frac{\ell^2}{mR^4}\,.$$

The approximation above comes from the fact that eq. (2), i.e., $f' = 0$, corresponds to a circular orbit, and so can be used only in the case of small deviations from a circle.

Let us denote the orbital angular frequency by $\omega_o = \dot{\theta} = \ell/mR^2$; then from eq. (3) we have

$$\omega_s = \sqrt{\omega_o^2 + 4\pi\rho G}\,.$$

First consider the approximation $\omega_s = \omega_o$. In the course of one orbital revolution, the radial distance varies sinusoidally between $R + s_{\max}$ and $R - s_{\max}$ and then back again. This is the same as the situation that led us to the derivation of an orbit

$$\frac{1}{r} = C[1 + e\cos(\theta - \theta_0)]$$

(see lecture notes or textbook). Since the total energy is $E < 0$, this corresponds to an elliptic orbit. An alternative "qualitative" argument uses the fact that a circular orbit is equivalent to a superposition of two perpendicular simple harmonic oscillators with equal frequency and amplitude. If the radial oscillation also has this same frequency, it is simply equivalent to changing the relative amplitude of the two perpendicular simple harmonic oscillators. The superposition of two perpendicular oscillators with different amplitude is well-known to be an ellipse.

Because of the small size of the ρ term, ω_s is just slightly larger than ω_o. Therefore the major (and minor) axes of the ellipse, instead of remaining fixed in space, rotate through a small angle for every orbital revolution. Since ω_s is a little larger ($\Rightarrow$ shorter period), the radial motion completes its cycle a little before θ has finished its revolution, resulting in a "backward" or opposite sign precession.