PHY 6/75101 November 1998

Classical Mechanics — Homework IX

This homework is due Wednesday December 2.

A. A simple plane pendulum has a mass m at the end of a string of length r. While the pendulum is in motion, the length of the string is changed at a constant rate $v_0 = \dot{r}$. Find the Lagrangian and the Hamiltonian, and consider whether or not $T + V$ and H are conserved. This is a rather famous problem discussed by Einstein, Lorentz and others at the 1911 Solvay Conference (see page 531 in Goldstein).

B. A simple pendulum consists of a massless rod of length R with a mass m at the bottom. The rod pivots on a hinge such that oscillation is constrained to lie in a plane. A synchronous electric motor rotates at a constant angular frequency ω and drives a vertical shaft, with the hinge attached to its lower end.
(1) Write down the Lagrangian and equation(s) of motion in a stationary coordinate system.
(2) What is the solution corresponding to stable small oscillations about the position of minimum energy, and what is the condition for this type of motion?
(3) What torque does the motor exert when the pendulum oscillates as above?
(4) What is the Hamiltonian in the standard (fixed) coordinate system, and in a rotating system attached to the shaft of the motor?
(5) Discuss whether or not $T+V$ and H are conserved in each of the two coordinate systems mentioned above.

C. Liouville's Theorem (see pages 426 – 428 in Goldstein) gives information about the statistical properties of systems containing a very large number of particles. It refers to the possible motion in phase space of systems that are identical to the system in question. Liouville's Theorem can be expressed by $\dot{D} = 0$, where D is the phase space density of possible systems in that region of phase space. There is no equivalent theorem that can be expressed in terms of quantities in configuration space, and so statistical mechanical problems are important examples where the Hamiltonian approach offers a solution while the Lagrangian approach does not.

Now consider the example of a beam of identical charged particles with momentum P, produced by an accelerator. Suppose that in the plane perpendicular to the incident direction, the beam initially has a uniformly-populated circular cross section of radius r_1 in configuration space, and a uniformly-populated circular cross section of radius p_1 in momentum space. A pair of quadrapole magnets with appropriate relative orientation can focus a beam of charged particles, i.e., can reduce the transverse radius from r_1 to r_2, where $r_2 < r_1$.
(1) What does Liouville's Theorem tell us about the consequences of this focusing operation?
(2) Suppose that the beam pipe has an internal radius R. At what maximum distance downstream from the focus must another focusing element be located in order to avoid some of the beam particles scraping the pipe?
(3) A collimator (an absorber with a hole in it) could also be used to produce a beam with radius r_2. Contrast the consequences of using a focusing element or a collimator.