Classical Mechanics — Homework V

This homework is due Monday October 26.

A. The *Voyager* missions to the outer planets of the solar system made use of a rare and especially favorable alignment of the outer planets in their orbits, which allowed the gravitational field of one planet to accelerate the spacecraft on its way to the next planet. This is sometimes called the "slingshot" effect. As a simplified illustration of this, consider the numerical example of a spacecraft which initially has a trajectory directly away from the sun, at escape velocity (with respect to the sun). It is timed to cross Jupiter's orbit just after Jupiter passes by. Assume that the timing has been adjusted so that the spacecraft enters an unbounded hyperbolic orbit about Jupiter with a 90° angle between the asymptotes. In other words, the spacecraft leaves the vicinity of Jupiter (the region of space where Jupiter's gravitational field influences the spacecraft's motion) with its velocity parallel to Jupiter's velocity. Calculate the final speed of the spacecraft and the fractional increase in its kinetic energy relative to the sun. [Hint: take Jupiter's orbit to be circular, and begin by finding the initial speed of the spacecraft, as viewed from Jupiter's rest frame. Then treat the spacecraft and Jupiter like an isolated two-body system.]

B. One side of earth's moon is always out of view on the earth, because the moon's orbital angular frequency is exactly the same as its rotational angular frequency. There are other examples in the solar system where an orbital angular frequency ω and a rotational angular frequency Ω, that one might expect to be uncorrelated, are in fact exactly synchronized (e.g., Mercury rotates exactly three times for every two of its orbits about the sun). The explanation is that ω and Ω generally start out being uncorrelated, but then tidal friction slowly dissipates the mechanical energy of the system in such a way as to synchronize ω and Ω.

Consider a small moon of mass m in a circular orbit of radius r about a large planet of mass M and radius R. As noted above, the moon's orbital angular frequency is ω. The easiest case to analyze is the less familiar one where the *planet* rotates with frequency Ω and the moon's rotation is neglected. Take the planet's rotation axis to be perpendicular to the plane of the moon's orbit. Show that if initially

$$\frac{\Omega}{\omega} < \frac{5mr^2}{8MR^2} + \tfrac{1}{4}\,,$$

then the system eventually stabilizes at $\Omega = \omega$.

C. An object dropped from rest at a height h above the earth's surface experiences a Coriolis deflection Δs_1 relative to where it would land if the earth's rotation were neglected. An object acted on by a vertical impulsive force such that it reaches a maximum height h before falling back to earth, experiences a Coriolis deflection Δs_2. Remarkably, the ratio $\Delta s_2/\Delta s_1$ is independent of all the parameters one might guess that it could depend on, e.g., h, latitude, g, etc. Find this ratio.

D. The circulation of air close to an intense high or low pressure region in the atmosphere follows a spiral that is well-approximated by a circle for purposes of calculating wind speeds.

(1) Find an expression connecting the speed $v(r)$ of wind at a distance r from the center of the high or low, in terms of the radial pressure gradient dP/dr. [Hints: centrifugal forces arising from the earth's rotation can be neglected. You should consider three terms, giving a quadratic equation in $v(r)$. It is helpful to write your equations in terms of $|dP/dr|$, so that the sign of each term appears explicitly.]

(2) Explain why destructive hurricane-force (typhoon-force) winds are associated only with low pressure (cyclones in the northern hemisphere).

(3) Estimate the maximum pressure gradient associated with a hurricane off Florida having sustained winds of 200 km per hour between $r = 20$ km and $r = 50$ km.