Sample Solutions — 1998 Classical Mechanics Homework Set V

V–A. We begin by calculating v_1, the initial speed of the spacecraft relative to the sun. We are told that $v_1 =$ escape velocity, so we use the standard expression (as calculated in HW I-B):

$$v_1(r) = \sqrt{2GM_{\rm sun}/r}\,.$$

Next, we need Jupiter's orbital speed v_J; in the approximation of a circular orbit, $v_J^2/r = GM_{\rm sun}/r^2$, leading to

$$v_J = \sqrt{GM_{\rm sun}/r}\,.$$

We are told that the direction of $\mathbf{v_1}$ is perpendicular to $\mathbf{v_J}$, so before the spacecraft comes close enough to Jupiter for Jupiter's gravitational field to have a significant effect, the spacecraft's speed relative to Jupiter is

$$v_1' = \sqrt{v_1^2 + v_J^2} = \sqrt{3GM_{\rm sun}/r}\,.$$

Next, we consider the relatively brief time interval when Jupiter dominates the motion of the spacecraft. This stage lasts only a tiny fraction of Jupiter's year, and it is a valid approximation to ignore the sun's gravity during this time interval. From the properties of motion under an inverse-square central force, we know that the spacecraft's path is a hyperbola with Jupiter at the focus. Angular momentum conservation then requires the asymptotic incoming speed v_1' to equal the asymptotic outgoing speed v_2'. To transform to the final speed of the spacecraft relative to the sun, we use the given information that the final direction is a tangent to Jupiter's orbit:

$$\begin{aligned} v_2 = v_2' + v_J &= \sqrt{\frac{GM_{\rm sun}}{r}}\,(\sqrt{3}+1) \\ &= \sqrt{\frac{(6.67\times10^{-11})(1.99\times10^{30})}{7.78\times10^{11}}}(\sqrt{3}+1) \\ &= 3.58\times10^4 \ \mathrm{m/s} \end{aligned}$$

The fractional increase in kinetic energy is

$$\left(\frac{v_2}{v_1}\right)^2 = \frac{(\sqrt{3}+1)^2}{2} = 3.73$$

Notice that the spacecraft receives a large boost in its kinetic energy without any fuel expenditure, and without changing its potential energy relative to Jupiter. The net effect of the "slingshot" process is that a small amount of Jupiter's orbital kinetic energy is transferred to the spacecraft.

V–B. The mechanical energy E of the planet-moon system consists of the gravitational potential energy, the orbital kinetic energy of the moon, and the rotational kinetic energy of the planet:

$$E = -\frac{GMm}{r} + \tfrac{1}{2}mr^2\omega^2 + \tfrac{1}{2}I\Omega^2 \,,$$

where $I = 2MR^2/5$ is the moment of inertia of the rotating planet (i.e., I for a solid sphere about any axis through its center). We want to find the condition for a stable minimum in E. The variables r and ω are not independent, so it is helpful to eliminate r and write E in the form $E(\omega,\Omega)$; to do this, we use the circular orbit condition $mr\omega^2 = GMm/r^2$, or

$$r = (GM)^{1/3}\omega^{-2/3}$$

to get

$$\begin{aligned} E &= -\frac{GMm}{r} + \tfrac{1}{2}\frac{GMm}{r} + \tfrac{1}{2}I\Omega^2 \\ &= \tfrac{1}{2}I\Omega^2 - \tfrac{1}{2}m(GM\omega)^{2/3} \,. \end{aligned} \tag{1}$$

Before we differentiate and set $dE = 0$, we need to incorporate the fact that while E decreases due to tidal friction, conservation of angular momentum, $\ell = I\Omega + mr^2\omega$ must be maintained. Again, we use the circular orbit condition to eliminate r; then conservation requires $d\ell = 0$:

$$\ell = I\Omega + m(GM)^{2/3}\omega^{-1/3} \,,$$

$$d\ell = I d\Omega - \tfrac{1}{3}m(GM)^{2/3}\omega^{-4/3}d\omega = 0$$

$$\Rightarrow \quad I d\Omega = \tfrac{1}{3}mr^2 d\omega \quad \text{or} \quad \frac{d\Omega}{d\omega} = \frac{mr^2}{3I} \tag{2}$$

Now we can return to take the derivative of eq. (1), then impose angular momentum conservation [eq. (2)], and set $dE = 0$:

$$\begin{aligned} dE &= I\Omega\, d\Omega - \tfrac{1}{3}m(GM)^{2/3}\omega^{-1/3}d\omega \\ &= \tfrac{1}{3}mr^2(\Omega - \omega)d\omega = 0 \,. \end{aligned}$$

Hence we know that there is a stationary point in the total mechanical energy when $\Omega = \omega$.

Next, we must differentiate again to show that this stationary point is a minimum. The condition we are asked to verify contains r, so there is no longer any particular reason to eliminate r.

$$\frac{d^2E}{d\omega^2} = \tfrac{2}{3}mr(\Omega - \omega)\frac{dr}{d\omega} + \tfrac{1}{3}mr^2\left(\frac{d\Omega}{d\omega} - 1\right)$$

Noting that

$$\frac{dr}{d\omega} = -\tfrac{2}{3}(GM)^{1/3}\omega^{-5/3} \;=\; -\frac{2r}{3\omega} \,,$$

and also using eq. (2), we can simplify the 2nd derivative as follows:

$$\begin{aligned}\frac{d^2E}{d\omega^2} &= -\frac{4mr}{9}(\Omega-\omega)\frac{r}{\omega}+\frac{mr^2}{3}\left(\frac{mr^2}{3I}-1\right)\\ &= \frac{mr^2}{9}\left(\frac{mr^2}{I}-3-\frac{4\Omega}{\omega}+4\right)\end{aligned}$$

To have a minimum in the mechanical energy at $\Omega=\omega$, the term in brackets must be positive, i.e.,

$$\frac{\Omega}{\omega}<\frac{mr^2}{4I}+\tfrac{1}{4}$$

Inserting $I=2MR^2/5$, we get the required condition

$$\frac{\Omega}{\omega}<\frac{5mr^2}{8MR^2}+\tfrac{1}{4}$$

If the initial conditions satisfy this inequality, then there is an allowed path in the Ω, ω, r parameter space along which the system can converge towards the stable point at $\Omega=\omega$, where the planet's day becomes synchronized with the moon's orbital period.

For the earth and our moon, the left side of the inequality is 27.4, while the right side is 28.0. Thus it appears that the condition for our moon to become a geosynchronous satellite is satisfied, but just barely so. Given that our calculation above is based on a number of simplifying approximations, it is risky to draw any firm conclusion from it.

In fact, more precise calculations of this type have been done by astronomers. The conclusion is that our moon will indeed become a geosynchronous satellite several billion years from now. By then, the earth's rotation will have slowed down to the point where a day will last 47 times longer than at present.

V–C. For this problem, we use the same axes and notation as Goldstein on p. 181, i.e., $\mathbf{x}$ points east, $\mathbf{y}$ points north, and $\mathbf{z}$ points up. Initially, assume we are at colatitude θ in the northern hemisphere.

$$\ddot{x}=-2\boldsymbol{\omega}\times\mathbf{v}=-2\omega_{\text{north}}v_{\text{vert}}=-2\omega\sin\theta(v_0-gt)\,.$$

Note that the Coriolis force also adds a correction to v_{vert}, but that is a second order small correction, and can be neglected. Thus, the vertical motion takes place under a constant acceleration g, and the horizontal displacement Δs takes place along an east-west line.

$$\dot{s}=-\omega\sin\theta(2v_0t-gt^2)+C$$

In both cases to be considered, $\dot{s}(t=0)=0$, therefore $C=0$.

Consider the deflection in the case of the body projected upward (case 2). It is convenient to consider this case first, since it is more general. The deflection can be written

$$\Delta s_2=\int_0^{\Delta t_2}\dot{s}dt$$

where $\Delta t_2 = 2v_0/g$ is the time in the air. Therefore

$$\begin{aligned}
\Delta s_2 &= \omega \sin\theta \left(\frac{g \Delta t_2^3}{3} - v_0 \Delta t_2^2 \right) \\
&= \omega \sin\theta \left(\frac{8v_0^3}{3g^2} - \frac{4v_0^3}{g^2} \right) \\
&= -\omega \sin\theta \left(\frac{4v_0^3}{3g^2} \right)
\end{aligned}$$

Now consider the other case, where the body is dropped from rest at the same maximum height. Then we set $v_0 = 0$ and the deflection becomes

$$\Delta s_1 = \int_0^{\Delta t_1} \dot{s} dt$$

where we ensure the same maximum height h by setting

$$\Delta t_1 = \frac{\Delta t_2}{2} = \frac{v_0}{g}$$

Then

$$\begin{aligned}
\Delta s_1 &= \omega \sin\theta \left(\frac{g \Delta t_1^3}{3} - 0 \right) \\
&= \omega \sin\theta \left(\frac{v_0^3}{3g^2} \right)
\end{aligned}$$

$$\Rightarrow \frac{\Delta s_2}{\Delta s_1} = -4$$

Since our positive s axis points east, the body projected upward is deflected to the west, whereas the body dropped from rest is deflected to the east.

In the southern hemisphere, $\boldsymbol{\omega}$ changes sign and so do the directions of deflection, but the ratio $\Delta s_2/\Delta s_1$ still remains equal to -4.

V–D. (1) The large-scale horizontal motion of the atmosphere arises from flow of air into regions of low atmospheric pressure, or out of regions of high pressure. The assumption of a circular wind pattern means we can work with an equation in r only (like a circular orbit under a central force).

We begin by finding the connection between the radial component of force F and the radial pressure gradient dP/dr. Consider a rectangular slab of air with its thin dimension along r. If we write the volume of this slab as Adr, then its mass is $dm = \rho A dr$, where ρ is the air density. Since pressure is defined as force per unit area normal to the force, we can write

$$\frac{dF}{dr} = A \frac{dP}{dr} = \frac{1}{\rho} \frac{dm}{dr} \frac{dP}{dr}$$

Consequently, the radial acceleration is

$$\frac{dF}{dm} = \frac{1}{\rho}\frac{dP}{dr}$$

There are three radial acceleration terms we need to consider:

- $(1/\rho)dP/dr$ as shown above.
- Centrifugal acceleration v^2/r due to rotation of air at speed v around the center of the region of high or low pressure.
- Coriolis acceleration $2v\omega_{\text{earth}} \sin\lambda$, where λ is the latitude.

In principle, any time ω_{earth} needs to be taken into account, there are two fictitious acceleration terms: a centrifugal acceleration and a Coriolis acceleration. However, in this particular instance, the centrifugal term in ω_{earth} is entirely negligible because of its small size and because it affects all parts of the circulating air mass equally.

The direction of air circulation is dictated by the Coriolis force. In the northern hemisphere, the Coriolis deflection relative to $\mathbf{v}$ is to the right (see the solution to Goldstein problem 4-23), and so air circulation is counterclockwise about a low and clockwise about a high. These directions are reversed in the southern hemisphere.

First we find the expression for $v_{\text{low}}(r)$, the wind speed surrounding a region of low pressure. For a low in *either* hemisphere, we know that $(1/\rho)|dP/dr|$ points IN and the Coriolis acceleration points OUT. A circular path $\Rightarrow a_{\text{in}} = a_{\text{out}}$, or

$$\frac{1}{\rho}\left|\frac{dP}{dr}\right| = \frac{v_{\text{low}}^2}{r} + 2\omega_\lambda v_{\text{low}} \tag{3}$$

where we use the shorthand notation $\omega_\lambda = \omega_{\text{earth}} \sin\lambda$. The roots of this quadratic in v_{low} are given by

$$\frac{v_{\text{low}}}{r} = -\omega_\lambda \pm \sqrt{\omega_\lambda^2 + \frac{1}{r\rho}\left|\frac{dP}{dr}\right|} \tag{4}$$

The positive sign in front of the square root must be the physical solution, since it leads to $v_{\text{low}} > 0$.

Next consider the expression for $v_{\text{high}}(r)$, the wind speed surrounding a region of high pressure. Now the acceleration due to the pressure gradient points OUT (it was IN for low pressure), and the Coriolis acceleration points IN (it was OUT for low pressure). Thus, we use a similar argument to obtain

$$\frac{v_{\text{high}}}{r} = \omega_\lambda - \sqrt{\omega_\lambda^2 - \frac{1}{r\rho}\left|\frac{dP}{dr}\right|} \tag{5}$$

(3) It is helpful (though not essential) to answer part (3) before part (2). From eq. (3), we see that $|dP/dr|_{\max}$ happens at the smallest radius where the wind speed v is a maximum, i.e., we substitute $r = 20$ km and $v_{\mathrm{low}} = 200$ km/hour. The mean latitude of Florida is 28° North. Inserting into eq. (3):

$$\begin{aligned} \frac{1}{\rho}\left|\frac{dP}{dr}\right|_{\max} &= \frac{200^2}{20}\left[\frac{\mathrm{km}}{\mathrm{hr}^2}\right] + 2\left(\frac{2\pi}{24}\sin 28^\circ\right)200\left[\frac{\mathrm{km}}{\mathrm{hr}^2}\right] \\ &= 2000 + 50\left[\frac{\mathrm{km}}{\mathrm{hr}^2}\right] \\ &= 2.05\times 10^3\,\frac{10^3}{60^4}\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] \end{aligned} \tag{6}$$

Inserting $\rho_{air} \sim 1.3$ kg/m^3, we get $|dP/dr|_{\max} \approx 0.2$ N/m^3.

(2) At the center of an exceptional low in the atmosphere, the pressure can drop to near 90% of normal, while an exceptional high can reach almost as far in the other direction. The very different behavior of air circulation in the two cases comes about because of the different orientation of the radial acceleration terms.

When a center of high or low pressure first forms, the air would tend to flow in a straight line directly towards the center of the low (or away from the center of the high) were it not for the Coriolis acceleration. This relatively small acceleration term is enough to break the symmetry and initiate a pattern of air circulation one way or the other. At this early stage, the spiral path of the air flow is not so close to a circle. However, the situation is very different just outside the eye of an intense hurricane, and here the air circulation is quite close to circular. Because of this circular path, the Coriolis term for a low pressure region points almost radially outward.

Eq. (6) above illustrates the fact that in a hurricane, the radial acceleration arising from the pressure gradient $(1/\rho)|dP/dr|$ is maybe 40 times larger than the Coriolis acceleration term. The wind speed increases with the square root of this large pressure gradient term [see eq. (4)], whereas the much weaker Coriolis term points in the opposite direction and effectively opposes the wind.

Conversely, when air circulates around a region of high pressure, the roles of the two acceleration terms are reversed [see eq. (5)]. It is the weak Coriolis acceleration, opposed by the pressure gradient, that causes the air to follow a curved path. For this reason, wind speeds normally do not exceed a few km/hr even for an exceptional high pressure region. Although the spiral air flow associated with high pressure generally is not very close to a circle, eq. (5) is still useful for indicating the approximate magnitude of the relevant parameters.