Sample Solutions — 1998 Classical Mechanics Homework Set VI

VI–A. See lecture notes.

VI–B. Let $x,\ y,\ z$ be a set of coordinates fixed to the plate, with its origin at the plate's center. The z axis is perpendicular to the plane of the plate. We are free to orient our x and y axes within the plane of the plate such that the vector $\boldsymbol{\omega}$ lies in the $x-z$ plane. (Another way to say it is that the vector $\boldsymbol{\omega}$ and the z axis define a plane that rotates as the plate rotates, and we choose our x axis so that it lies in this plane.)

The circular disc of radius R has moments $I_{xx} = I_{yy} = mR^2/4$ about its center. Therefore, using the perpendicular axis theorem,

$$I_{zz} = I_{xx} + I_{yy} = \tfrac{1}{2}mR^2.$$

The vector $\boldsymbol{\omega}$ and its derivative can be written in this coordinate system as

$$\begin{aligned} \boldsymbol{\omega} &= (\omega \sin\psi,\ 0,\ \omega\cos\psi) \\ \dot{\boldsymbol{\omega}} &= (0,\ 0,\ 0) \end{aligned}$$

(since ω and ψ are both constant).

Euler's equations in the $x,\ y,\ z$ frame are

$$\begin{aligned} I_{xx}\dot{\omega}_x - \omega_y\omega_z(I_{yy} - I_{zz}) &= N_x \\ I_{yy}\dot{\omega}_y - \omega_z\omega_x(I_{zz} - I_{xx}) &= N_y \\ I_{zz}\dot{\omega}_z - \omega_x\omega_y(I_{xx} - I_{yy}) &= N_z \end{aligned}$$

Thus, $N_x = N_z = 0$ and

$$\begin{aligned} N_y &= -\omega^2 \sin\psi\ \cos\psi\, mR^2(\tfrac{1}{2} - \tfrac{1}{4}) \\ &= -\tfrac{1}{4}mR^2\omega^2 \sin\psi\ \cos\psi \end{aligned}$$

This torque lies in the plane of the plate, and its direction is at 90° to the plane defined by $\mathbf{z}$ and $\boldsymbol{\omega}$. Thus, the torque direction must rotate as the disc rotates.

V–C. (1) Let ω_1, ω_2 and ω_3 be the principal axes corresponding to the moments $I_1,\ I_2$ and I_3, respectively. Then, Euler's equations can be written in the form

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= 0 \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= 0 \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= 0 \end{aligned}$$

First consider the case where the spin initially is close to being aligned with the ω_1 axis. Then the product $\omega_2\omega_3$ is very small and the first Euler eq. $\Rightarrow$ ω_1 is approximately constant. Therefore

$$\dot{\omega}_2 = \frac{\omega_3\omega_1(I_3 - I_1)}{I_2}$$

and the approximate constancy of ω_1 allows us to write

$$\begin{aligned} \ddot{\omega}_2 &\approx \dot{\omega}_3 \frac{\omega_1(I_3 - I_1)}{I_2} \\ &\approx \omega_2 \left[\frac{\omega_1^2(I_3 - I_1)(I_1 - I_2)}{I_2 I_3}\right] . \end{aligned} \tag{1}$$

The same reasoning also implies

$$\ddot{\omega}_3 \approx \omega_3 \left[\frac{\omega_1^2(I_1 - I_2)(I_3 - I_1)}{I_3 I_2}\right] . \tag{2}$$

Note that the terms in square brackets are both negative, since $I_1 < I_2 < I_3$. Therefore, eqs. (1) and (2) tell us that simple harmonic oscillation takes place in the magnitudes of ω_2 and ω_3, and the motion remains predominantly a rotation about the ω_1 axis.

Next, consider the case where the spin is almost along ω_3. This can be described by exactly the same equations if we just switch our ordering convention to $I_1 > I_2 > I_3$. In this case, the square bracket terms are still negative and so the conclusion is the same.

Finally, consider the case where the spin is almost along ω_2. Now we find that initially,

$$\begin{aligned} \ddot{\omega}_1 &\approx \omega_1 \left[\frac{\omega_2^2(I_2 - I_3)(I_1 - I_2)}{I_1 I_3}\right] \\ \text{and} \qquad \ddot{\omega}_3 &\approx \omega_3 \left[\frac{\omega_2^2(I_1 - I_2)(I_2 - I_3)}{I_3 I_1}\right] . \end{aligned}$$

Now the terms in square brackets are both positive, which implies a diverging solution with a growing magnitude of angular frequency in the other two directions ω_1 and ω_3.

(2) When $I_1 < I_2 = I_3$, we see that the square bracket terms either remain negative or become zero. Therefore, unstable rotation is not observed when $I_2 = I_3$.

VI–D. An obvious, but somewhat lengthy, approach to this problem is to calculate directly the moment of inertia I_1 of a cube of sides ℓ about its space diagonal; the answer, as obtained in an example worked in class is $I_1 = M\ell^2/6$. Then do the same for an axis through the center of two opposite faces; the answer again is $I_0 = M\ell^2/6$. Therefore, $I_1 = I_0$.

A shorter method is as follows. Consider a Cartesian coordinate system with its origin at the center of the cube and axes parallel to the sides. We are told that $I_{xx} = I_{yy} = I_{zz} = I_0$. Also, the off-diagonal inertia tensor elements like $I_{xy},\ I_{yz},\ I_{zx}$ are all zero. This can be seen either from direct calculation, or just by noting from symmetry arguments that the **x**, **y** and **z** axes are principal axes.

In general, once the full inertia tensor **I** is known, the moment of inertia I about any axis

$$\hat{\mathbf{n}} = (\alpha\mathbf{i},\ \beta\mathbf{j},\ \gamma\mathbf{k})$$

is obtained from [see lecture notes or eqs. (5-19) and (5-32) in Goldstein]

$$\begin{aligned} I &= \hat{\boldsymbol{n}} \cdot \mathbf{I} \cdot \hat{\boldsymbol{n}} \\ I &= \alpha^2 I_{xx} + \beta^2 I_{yy} + \gamma^2 I_{zz} + 2\alpha\beta I_{xy} + 2\beta\gamma I_{yz} + 2\gamma\alpha I_{zx} \\ &= I_0(\alpha^2 + \beta^2 + \gamma^2)\,. \end{aligned} \tag{3}$$

We are interested in the case where the unit vector $\hat{\mathbf{n}} = \hat{\mathbf{n}}_1$ is along a space diagonal. Symmetry tells us that the vector pointing from the cube's center to any corner will give the same result. Therefore, let us take the corner where all coordinates are positive:

$$\hat{\mathbf{n}}_1 = \frac{1}{\sqrt{3}}(\mathbf{i},\ \mathbf{j},\ \mathbf{k})\,.$$

Thus $\alpha^2 + \beta^2 + \gamma^2 = 1$ in eq. (3) above and consequently $I_1 = I_0$.

VI–E. In all three cases, the axis of torsional oscillation passes through the center of the cube. As illustrated by problem **D** above, the moment of inertia of the cube about all three axes is the same. In fact, the rotational inertia of a cube about *any* axis through its center is fully described by the inertia tensor $I_{jk} = \frac{1}{6}M\ell^2\delta_{jk}$. This is identical to the inertia tensor for a sphere of radius $R = \sqrt{(5/12)}\,\ell$. In the language of inertia ellipsoids (see Goldstein page 202), the inertia ellipsoid is a sphere for any body where all three principal moments are equal. Thus, the periods of torsional oscillation are the same in all three cases.