PHY 6/75101 December 1998

Classical Mechanics — Homework X

This homework is due Friday December 11.

A. Consider a particle of mass m which moves in two dimensions in a potential well. Let x_1 and x_2 be Cartesian coordinates with their origin at the point of lowest potential. The well is termed *isotropic* if the potential is the same for all directions in the (x_1, x_2) plane (e.g., for small oscillations, an isotropic potential must be a paraboloid of revolution).
(1) First consider an *anisotropic* oscillator where the potential is $V = \frac{1}{2}k(x_1^2 + x_2^2) + k' x_1 x_2$. Find the eigenfrequencies and normal coordinates of this system, and describe each normal mode of vibration.
(2) Use a qualitative physics-based argument to write down two independent constants of the motion. Verify your choice using the Poisson bracket equation

$$\dot{u} = [u, H] + \frac{\partial u}{\partial t},$$

where $u = u(q, p, t)$, and H is the Hamiltonian.
(3) The oscillator becomes *isotropic* if $k' = 0$. Again use a qualitative physics-based argument to write down an additional independent constant of the motion. Again verify your choice using the Poisson bracket equation above.

B. (1) Verify the Poisson bracket relationship

$$[L_i, L_j] = \epsilon_{ijk} L_k$$

among the Cartesian components of angular momentum for a spherical pendulum (see eq. 9-128 in Goldstein).
(2) Likewise, verify $[p_\theta, p_\phi] = 0$ for the spherical pendulum.
(3) The mathematical machinery of Poisson brackets evidently tells us that some perpendicular momentum components are valid canonical momenta (e.g., p_θ and p_ϕ), while others are not (e.g., the Cartesian components of angular momentum above). Explain the physics behind this.

C. Suppose that a system with a time-independent Hamiltonian $H_0(q, p)$ is modified such that the Hamiltonian becomes

$$H = H_0(q, p) - \epsilon q \sin \omega t,$$

where ϵ and ω are known constants.
(1) Apply Hamilton's canonical equations of motion to the modified system.
(2) Use a Canonical Transformation generating function F_2 to find a new Hamiltonian K and new canonical variables Q, P for the modified system such that $K(Q, P) = H_0(q, p)$.
(3) Verify that the transformed quantities K, Q, P satisfy Hamilton's canonical equations of motion.
(4) Suggest one possible physical interpretation for the term which modifies H_0.

D. A projectile is fired at an angle α measured from the horizontal, with initial velocity v_0. Take $x = z = 0$ at time $t = 0$. Use the Hamilton-Jacobi technique to determine $x(t)$, $z(t)$ and $z(x)$.