Sample Solutions — 1998 Classical Mechanics Homework Set X

X–A. (1) Because of the $x_1 x_2$ term in V, the $\mathbf{V}$ tensor will have nonzero off-diagonal terms and so x_1 and x_2 cannot be normal coordinates. The given potential V is the simplest anisotropic form, and we might guess that the equipotential contours have the shape of an ellipse. (Nothing that follows depends on knowing that the shape is elliptical.) As long as we retain Cartesian coordinates, the only possible way to transform to a set of normal coordinates is to apply a rotation. Because of the symmetry of V in x_1 and x_2, a 45° rotation is a good bet to try first, i.e., define

$$\zeta_1 = \frac{1}{\sqrt{2}}(x_1 + x_2) \qquad \text{and} \qquad \zeta_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

$$\Rightarrow \quad x_1 = \frac{1}{\sqrt{2}}(\zeta_1 + \zeta_2) \qquad \text{and} \qquad x_2 = \frac{1}{\sqrt{2}}(\zeta_1 - \zeta_2)$$

Then the kinetic energy $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ and the potential energy V become

$$\begin{aligned} T &= \tfrac{1}{4}m[(\dot{\zeta}_1 + \dot{\zeta}_2)^2 + (\dot{\zeta}_1 - \dot{\zeta}_2)^2] \\ &= \tfrac{1}{2}m(\dot{\zeta}_1^2 + \dot{\zeta}_2^2)\,. \\ V &= \tfrac{1}{4}k[(\zeta_1 + \zeta_2)^2 + (\zeta_1 - \zeta_2)^2] + \tfrac{1}{2}k'(\zeta_1^2 - \zeta_2^2) \\ &= \tfrac{1}{2}k(\zeta_1^2 + \zeta_2^2) + \tfrac{1}{2}k'(\zeta_1^2 - \zeta_2^2) \\ &= \tfrac{1}{2}(k + k')\zeta_1^2 + \tfrac{1}{2}(k - k')\zeta_2^2\,. \end{aligned}$$

Now we see that $V(\zeta_1, \zeta_2)$ has the desired form leading to a diagonal tensor $\mathbf{V}$, and so ζ_1 and ζ_2 are normal coordinates. The system executes Simple Harmonic Motion in each normal coordinate ζ_1 and ζ_2, and the angular frequencies of oscillation are

$$\omega_1 = \sqrt{\frac{k + k'}{m}} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{k - k'}{m}}\,.$$

The first normal mode is SHM with angular frequency ω_1 along the "steepest" direction inside the potential well (the minor axis of the ellipse), while the other normal mode is SHM with angular frequency ω_2 along the "least steep" direction inside the potential well (the major axis of the ellipse). This identification of normal modes of course presupposes that k' is positive and that $k' < k$.

(2) The total mechanical energy $E = T + V$ of any Simple Harmonic Oscillator is a constant of the motion. In this example, we might guess the energy E_i associated with each of the two normal modes to be conserved separately (and we verify this below). For an SHO, the conservative potential allows us to write the Hamiltonian as $H = E_1 + E_2$. Also

$$E_1 = \frac{p_1^2}{2m} + \tfrac{1}{2}(k + k')\zeta_1^2 \qquad \text{and} \qquad E_2 = \frac{p_2^2}{2m} + \tfrac{1}{2}(k - k')\zeta_2^2$$

where the canonical momenta are $p_i = \partial L_i / \partial \dot{\zeta}_i = m\dot{\zeta}_i$, and L is the Lagrangian $T - V$.

Now we test to see if we have guessed correctly that E_1 and E_2 are each conserved; first consider E_1:

$$\begin{aligned}
\dot{E}_1 &= [E_1, H] + \frac{\partial E_1}{\partial t} \\
&= [E_1, E_1 + E_2] \\
&= [E_1, E_1] + [E_1, E_2] \\
&= [E_1, E_2] \\
&= \frac{\partial E_1}{\partial \zeta_i}\frac{\partial E_2}{\partial p_i} - \frac{\partial E_1}{\partial p_i}\frac{\partial E_2}{\partial \zeta_i} \\
&= \frac{\partial E_1}{\partial \zeta_1}\frac{\partial E_2}{\partial p_1} + \frac{\partial E_1}{\partial \zeta_2}\frac{\partial E_2}{\partial p_2} - \frac{\partial E_1}{\partial p_1}\frac{\partial E_2}{\partial \zeta_1} - \frac{\partial E_1}{\partial p_2}\frac{\partial E_2}{\partial \zeta_2} \\
&= 0\,.
\end{aligned}$$

We could follow the same steps to show that $\dot{E}_2 = 0$, or else we can argue that since the total energy $H = E_1 + E_2$ is conserved for sure, then $E_2 = H - E_1$ must be conserved.

(3) If the potential becomes rotationally symmetric (isotropic), then the polar angle of the particle in the $x_1 - x_2$ plane becomes cyclic. The new conserved momentum is the angular momentum $\ell_3 = m(\zeta_1 p_2 - \zeta_2 p_1)$ about the x_3 axis, where x_3 is perpendicular to both x_1 and x_2.

Now the Hamiltonian becomes

$$H = \tfrac{1}{2}\left(\frac{p_1^2}{m} + \frac{p_2^2}{m} + k\zeta_1^2 + k\zeta_2^2\right).$$

We test whether or not ℓ_3 is a constant of the motion using the given Poisson bracket formula:

$$\begin{aligned}
\dot{\ell}_3 &= [\ell_3, H] + \frac{\partial \ell_3}{\partial t} \\
&= [\ell_3, H] \\
&= \frac{\partial \ell_3}{\partial \zeta_i}\frac{\partial H}{\partial p_i} - \frac{\partial \ell_3}{\partial p_i}\frac{\partial H}{\partial \zeta_i} \\
&= p_2 p_1 - p_1 p_2 + mk\zeta_2\zeta_1 - mk\zeta_1\zeta_2 \\
&= 0\,.
\end{aligned}$$

This verifies that the angular momentum ℓ_3 is conserved.

X–B. (1) We have considered the spherical pendulum in several previous problems. Here, we do not go back to the very beginning, but start from our previous expression for the Hamiltonian in the case of a mass m and pendulum length R:

$$H = T + V = \frac{1}{2m}\left(\frac{p_\theta^2}{R} + \frac{p_\varphi^2}{R^2\sin^2\theta}\right) - mgR\cos\theta\,,$$

$$\text{where} \qquad p_\theta = mR^2\dot\theta \qquad \text{and} \qquad p_\varphi = mR^2\sin^2\theta\,\dot\varphi\,.$$

We are asked to verify an equation whose first component is $L_x = [L_y, L_z]$, with the other two following in cyclic order. (Do not be confused by the present notation, taken from Goldstein, where L_i stands for a Cartesian component of angular momentum.) To evaluate these components, we must first write down the Cartesian spatial and momentum coordinates in terms of our chosen generalized coords and momenta $\theta, \varphi, p_\theta$ and p_φ. The three Cartesian spatial coordinates are:

$$(x, y, z) = R\,(\sin\theta\cos\varphi,\ \ \sin\theta\sin\varphi,\ \ \cos\theta)$$

and the first of the three Cartesian momentum components is:

$$\begin{aligned} p_x = m\dot x &= mR(\cos\theta\cos\varphi\,\dot\theta - \sin\theta\sin\varphi\,\dot\varphi) \\ &= \frac{p_\theta}{R}\cos\theta\cos\varphi - \frac{p_\varphi\sin\varphi}{R\sin\theta}\,. \end{aligned}$$

Likewise, the other two components are

$$\begin{aligned} p_y = m\dot y &= \frac{p_\theta}{R}\cos\theta\cos\varphi - \frac{p_\varphi\cos\varphi}{R\sin\theta}\,; \\ p_z = m\dot z &= -\frac{p_\theta}{R}\sin\theta\,. \end{aligned}$$

Now we can evaluate the Cartesian angular momentum components:

$$\begin{aligned} L_x &= yp_z - zp_y \\ &= R\sin\theta\sin\varphi(-\frac{p_\theta}{R}\sin\theta) - R\cos\theta\left(\frac{p_\theta}{R}\cos\theta\cos\varphi - \frac{p_\varphi\cos\varphi}{R\sin\theta}\right) \\ &= -p_\theta\sin\varphi - p_\varphi\cot\theta\cos\varphi\,. \end{aligned}$$

Likewise, the next component and the last component are

$$\begin{aligned} L_y &= zp_x - xp_z = p_\theta\cos\varphi - p_\varphi\cot\theta\sin\varphi\,, \\ L_z &= p_\varphi\,. \end{aligned}$$

Next, we evaluate the Poisson bracket term

$$[L_y, L_z] = \frac{\partial L_y}{\partial\theta}\frac{\partial L_z}{\partial p_\theta} + \frac{\partial L_y}{\partial\varphi}\frac{\partial L_z}{\partial p_\varphi} - \frac{\partial L_y}{\partial p_\theta}\frac{\partial L_z}{\partial\theta} - \frac{\partial L_y}{\partial p_\varphi}\frac{\partial L_z}{\partial\varphi}$$

The second partial derivative in each of these terms is zero except for the second term where $\partial L_z/\partial p_\varphi = 1$, giving us

$$\begin{aligned}[L_y, L_z] &= \frac{\partial L_y}{\partial \varphi} \\ &= -p_\theta \sin\varphi - p_\varphi \cot\theta \cos\varphi \\ &= L_x\,.\end{aligned}$$

The above verifies the first component of the equation given in the problem. Since the angular momentum components are defined in strictly cyclic order, the other two equations to be verified directly follow by cyclic permutation of $[L_y, L_z] = L_x$. Alternatively, we could just rerun the above type of proof for the other two equations.

(2) We have shown in general that $[p_i, p_j] = 0$ for independent generalized momenta, and p_θ and p_φ are clearly examples of such momenta. While this is sufficient to prove the assigned problem, we can still go through the procedural steps to prove $[p_\theta, p_\varphi] = 0$ for a spherical pendulum:

$$\begin{aligned}[p_\theta, p_\varphi] &= \frac{\partial p_\theta}{\partial \theta}\frac{\partial p_\varphi}{\partial p_\theta} + \frac{\partial p_\theta}{\partial \varphi}\frac{\partial p_\varphi}{\partial p_\varphi} - \frac{\partial p_\theta}{\partial p_\theta}\frac{\partial p_\varphi}{\partial \theta} - \frac{\partial p_\theta}{\partial p_\varphi}\frac{\partial p_\varphi}{\partial \varphi} \\ &= 0\,.\end{aligned}$$

(3) Two independent coordinates such as θ and φ are sufficient to fully specify the rotational orientation of a rigid body in three dimensions. For that reason, it is impossible for the three Cartesian components of angular momentum to be independent of each other, and so we find that the Poisson brackets $[L_i, L_j]$ do not vanish.

There is useful further discussion of this and related questions on page 419 in Goldstein. You are especially encouraged to read the footnote on that page, where the connection with quantum mechanics is discussed.

X–C. (1) Hamilton's canonical equations of motion are

$$\begin{aligned}\dot q = \frac{\partial H}{\partial p} &= \frac{\partial H_0}{\partial p} \\ \text{and} \qquad \dot p = -\frac{\partial H}{\partial q} &= -\frac{\partial H_0}{\partial q} + \epsilon \sin\omega t\,.\end{aligned}$$

This is as far as we can go without knowing the explicit form of H_0.

(2) The generating function $F_2(q, P, t)$ has the properties

$$p = \frac{\partial F_2}{\partial q}, \qquad Q = \frac{\partial F_2}{\partial P}, \qquad \text{and} \qquad K = H + \frac{\partial F_2}{\partial t} \tag{1}$$

The third relationship above tells us that

$$\frac{\partial F_2}{\partial t} = K - H = \epsilon q \sin\omega t\,. \tag{2}$$

It is not obvious how to find F_2, because the partial derivative above contains no information about the q and P dependence of F_2. Our only information is that after the canonical transformation, there is no explicit time dependence and the transformed Hamiltonian is $K = H_0$. At this point, we take a (wild?) guess that the q, P dependence has the same form as the identity canonical transformation $F_2 = q_i P_i$, with an added term in t as required to satisfy eq. (2). Thus our guess is:

$$F_2 = qP - \frac{\epsilon q}{\omega}\cos\omega t\,.$$

Then eq. (1) gives the canonical transformation equations

$$\begin{aligned}
P &= p + \frac{\epsilon}{\omega}\cos\omega t \\
Q &= q \\
\text{and}\quad K &= H(q,p,t) + \epsilon q\sin\omega t \\
&= H_0 - \epsilon q\sin\omega t + \epsilon q\sin\omega t \\
&= H_0\,,
\end{aligned}$$

where the transformed Hamiltonian K can be written in the form

$$\begin{aligned}
K(Q,P) &= H_0(q,p) \\
&= H_0(Q, P - \frac{\epsilon}{\omega}\cos\omega t)\,.
\end{aligned}$$

The above of course verifies that our guess for F_2 was correct.

(3) To verify that Hamilton's equations of motion are still valid after the canonical transformation, we take

$$\frac{\partial K}{\partial P} = \frac{\partial H_0}{\partial p}\frac{\partial p}{\partial P} = \frac{\partial H_0}{\partial p} = \dot{q} = \dot{Q}$$

$$-\frac{\partial K}{\partial Q} = -\frac{\partial H_0}{\partial q}\frac{\partial q}{\partial Q} = -\frac{\partial H_0}{\partial q} = \dot{p} - \epsilon\sin\omega t = \dot{P}\,.$$

The above proves that the transformation is indeed canonical, and that Hamilton's equations of motion are satisfied.

(4) The term $\epsilon q\sin\omega t$ is subtracted from the Hamiltonian and it must have the dimensions of energy. Thus the term $\epsilon\sin\omega t$ has the form of a generalized force, oscillating with angular frequency ω. The particle in the modified system thus experiences a force whose variation in phase space is not known; however, the force is time dependent, with a sinusoidal variation according to $\epsilon \sin \omega t$.

X–D. To avoid possible confusion between the given initial angle of the projectile, α, and the constants of integration which arise in the Hamilton-Jacobi method, we only substitute the angle α at the end of the solution. For a projectile of mass m in a uniform gravitational field, the Hamiltonian is

$$H = \frac{p_x^2 + p_z^2}{2m} + mgz\,.$$

This is an example where coordinates are separable and there is just one non-cyclic coordinate: z.

$$\begin{aligned}
x \text{ is cyclic} \quad &\Rightarrow \quad p_x = \alpha_x \\
t \text{ is cyclic} \quad &\Rightarrow \quad E = \alpha_1 = \frac{\alpha_x^2}{2m} + \frac{1}{2m}\left(\frac{\partial W_z}{\partial z}\right)^2 + mgz \\
&\Rightarrow \quad \frac{\partial W_z}{\partial z} = (2m\alpha_1 - \alpha_x^2 - 2m^2 gz)^{1/2} \\
\text{or} \quad W_z &= -\frac{2}{6m^2 g}(2m\alpha_1 - \alpha_x^2 - 2m^2 gz)^{3/2}\,.
\end{aligned}$$

From eq. [77] in our notes (or Goldstein 10-39), we know

$$\begin{aligned}
W &= W_z + \sum_{i=2}^{n} \alpha_i q_i \\
&= W_z + \alpha_x x
\end{aligned}$$

Also, refer to eq. (10-28b) on p. 446:

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} \quad \text{and} \quad \beta_x = \frac{\partial W}{\partial \alpha_x}\,,$$

$$\Rightarrow \quad t + \beta_1 = -\frac{1}{mg}\sqrt{2m\alpha_1 - \alpha_x^2 - 2m^2 gz} \tag{3}$$

$$\text{and} \quad \beta_x = x + \frac{\alpha_x}{m^2 g}\sqrt{2m\alpha_1 - \alpha_x^2 - 2m^2 gz}\,. \tag{4}$$

Now the physics of the problem is largely done, and it just remains to do some algebra to solve for the motion $x(t)$, $z(t)$ and the trajectory.

At $t = 0$, we are given that $x = z = 0$. Let $\dot{x}_0$ and $\dot{z}_0$ be the horizontal and vertical components, respectively, of the initial velocity v_0. Then eq. (3) above yields

$$\begin{aligned}
\beta_1 &= -\frac{1}{mg}\sqrt{2m(\tfrac{1}{2}mv_0^2) - m^2\dot{x}_0^2} \\
&= -\frac{1}{mg}m\dot{z}_0 = -\frac{\dot{z}_0}{g}
\end{aligned}$$

$$\text{Eq. (4)} \quad \Rightarrow \quad \beta_x = 0 + \frac{m\dot{x}_0^2}{m^2 g} m\dot{z}_0 = \frac{\dot{x}_0 \dot{z}_0}{g}$$

Now we solve Eq. (3) to obtain $z(t)$:

$$\begin{aligned} t - \frac{\dot{z}_0}{g} &= -\frac{1}{mg}\sqrt{(m\dot{z}_0)^2 - 2m^2 g z} \\ t^2 - \frac{2t\dot{z}_0}{g} + \frac{\dot{z}_0^2}{g^2} &= \frac{1}{m^2 g^2}(m^2 \dot{z}_0^2 - 2m^2 g z) \\ &= \frac{\dot{z}_0^2}{g^2} - \frac{2z}{g} \end{aligned}$$

$$\Rightarrow \quad z(t) = -\frac{g}{2}t^2 + \dot{z}_0 t = -\frac{g}{2}t^2 + v_0 t \sin\alpha$$

Also, cyclic x and initial conditions give us

$$x(t) = \frac{\alpha_x}{m} t = v_0 t \cos\alpha$$

Finally, we determine the trajectory $z(x)$ by solving eq. (4):

$$\frac{\dot{x}_0 \dot{z}_0}{g} = x + \frac{m\dot{x}_0}{m^2 g}\sqrt{m^2 \dot{z}_0^2 - 2m^2 g z}$$

$$\begin{aligned} \left(\frac{\dot{x}_0 \dot{z}_0}{g}\right)^2 - 2x\left(\frac{\dot{x}_0 \dot{z}_0}{g}\right) + x^2 &= \frac{m^2 \dot{x}_0^2}{m^4 g^2}(m^2 \dot{z}_0^2 - 2m^2 g z) \\ &= \left(\frac{\dot{x}_0 \dot{z}_0}{g}\right)^2 - \frac{2\dot{x}_0^2 z}{g} \end{aligned}$$

i.e., the trajectory is a parabola of the form

$$\frac{2\dot{x}_0^2 z}{g} - 2x\frac{\dot{x}_0 \dot{z}_0}{g} + x^2 = 0\,.$$