Sample Solutions — Classical Mechanics Homework Set III

A. *Goldstein Exercise 2–9:*

If a generalized force Q is not included in the Lagrangian of a system, then Lagrange's Eq. can be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

Let Q_j be the impulsive force; then the potential term in L contains the effect of all remaining forces.

$$\int_{\Delta t} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) dt - \int_{\Delta t} \frac{\partial L}{\partial q_j} dt = \int_{\Delta t} Q_j dt = S_j$$

The quantity $\partial L/\partial q_j$, which accounts for any non-impulsive forces, is finite and so the second term must vanish as $\Delta t \to 0$. Then

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right)_f - \left(\frac{\partial L}{\partial \dot{q}_j}\right)_i = S_j$$

B. *Goldstein Exercise 2–18:*

First of all, it is clear that V and hence L is not a function of time, so energy is conserved in all cases (a) through (g) below. (No points are deducted if you failed to mention energy, since it is not obvious that it comes within the scope of the question).

(a) $V = V(z)$ and $V \neq V(x, y)$. Thus,
translational symmetry in the $x - y$ plane $\Rightarrow$ p_x, p_y conserved;
rotational symmetry about the z axis $\Rightarrow$ angular momentum ℓ_z conserved.

(b) Translational symmetry along the x direction $\Rightarrow$ p_x conserved.

(c) Translational symmetry along the z direction $\Rightarrow$ p_z conserved;
rotational symmetry about the z axis $\Rightarrow$ ℓ_z conserved.

(d) Rotational symmetry about the z axis $\Rightarrow$ ℓ_z conserved.

(e) Translational symmetry along the z direction $\Rightarrow$ p_z conserved.

(f) Rotational symmetry about the z axis $\Rightarrow$ ℓ_z conserved.

(g) I will try to explain each step in more detail than usual, because there has been an unusual number of questions and confusion about this part. In general, when we identify the symmetry in the system, we are identifying one or more coordinates q_k such that $V \neq V(q_k)$. In other words, $\partial L/\partial q_k = 0$. In parts (a) through (f), the cyclic coordinate or coordinates can readily be identified by inspection.

In case (g), the q_k such that $V \neq V(q_k)$ must follow a helical pattern that tracks the symmetry of the helical mass distribution. Let us consider cylindrical coordinates (r, θ, z) where $r = 0$ is the axis of the "solenoid", even though we realize from the outset that none of these three coordinates is the q_k we are looking for. Then we set

$$\delta L = 0 = \frac{\partial L}{\partial r}\delta r + \frac{\partial L}{\partial \theta}\delta\theta + \frac{\partial L}{\partial z}\delta z \ .$$

The condition $\delta L = 0$ leads us to a set of equations which are the conditions for staying on an equipotential path, which must be a helix at a constant distance from the axis of the solenoid. Thus, we require $r =$ constant ($\delta r = 0$) and $\delta z/D = \delta\theta/2\pi$, where D is the pitch (z distance between turns). It is not necessary for us to end up with an expression $q_k =$ whatever. We know that $\partial L/\partial q_k = 0$ if we satisfy our two equations above, and that's all we need. Then

$$\left(\frac{\partial L}{\partial \theta} + \frac{\partial L}{\partial z}\frac{\partial z}{\partial \theta}\right)\delta\theta = 0$$

or

$$\frac{\partial L}{\partial \theta} = -\frac{D}{2\pi}\frac{\partial L}{\partial z}$$

From Lagrange's eq. we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z}$$

Eliminating $\partial L/\partial\theta$ and $\partial L/\partial z$, we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -\frac{D}{2\pi}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)$$

Hence the conserved conjugate momentum is

$$p_\theta + \frac{D}{2\pi}p_z \qquad \text{or} \qquad \ell_z + \frac{D}{2\pi}p_z$$

C. *Goldstein Exercise 2–21:*

For background material related to this problem, see lecture notes near [16] and [17], or pages 47 through 49 in Goldstein. The surface supplies a constraint $\sigma = 0$ (by definition holonomic) and we want to end up with the associated force of constraint appearing explicitly in an expression for the energy of the particle. Therefore, we use the Lagrange Multiplier method. In cartesian coordinates,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \sum_l \lambda_l a_{lx} \,,$$

and likewise for y and z. In this problem, we have just one constraint — the reaction force that always points normal to the surface — and so using eq. [17], the right-hand side is

$$\lambda_1 a_{1x} = \lambda_1 \frac{\partial \sigma}{\partial x} \,.$$

In vector notation, we have

$$\dot{\mathbf{p}} + \nabla V = \lambda_1 \nabla \sigma \,.$$

Our standard procedure before integrating an equation of this form is to multiply across by velocity:

$$m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \nabla V \cdot \dot{\mathbf{r}} = \lambda_1 \nabla \sigma \cdot \dot{\mathbf{r}}$$

The left-hand side of the above is

$$\frac{d}{dt}\left(\tfrac{1}{2} m \dot{\mathbf{r}}^2\right) + \frac{dV}{dt} = \dot{E}$$

i.e., the rate of change of the total energy $E = T + V$. To see what the σ term represents, note that

$$\sigma(\mathbf{r}, t) = 0 \quad \Rightarrow \quad \dot{\sigma} = 0 = \nabla \sigma \cdot \dot{\mathbf{r}} + \frac{\partial \sigma}{\partial t} \,.$$

Thus our final result is

$$\dot{E} = -\lambda_1 \frac{\partial \sigma}{\partial t}$$

and we see that the energy changes with time unless $\partial \sigma / \partial t = 0$ due to the surface not moving. Remember that E is the energy of the particle in the potential V, and the force of constraint from the surface is external to this system. If the particle is a baseball and the surface is a moving bat, we should not be surprised to find that the energy of the ball changes.