CLASSICAL ELECTRODYNAMICS II

Homework Set 7

November 13, 2009

1. A 4-vector is defined as a set of four real numbers that transforms the same as the set of coordinates $(ct, x, y, z) = (ct, \mathbf{r})$. The Lorentz transformation equations for an arbitrary 4-vector $A = (A_0, \mathbf{A})$ can be written as

$$\begin{aligned} A_0' &= \gamma(A_0 - \boldsymbol{\beta} \cdot \mathbf{A}) \ , \\ \mathbf{A}_{\|}' &= \gamma(\mathbf{A}_{\|} - \boldsymbol{\beta} A_0) \ , \\ \mathbf{A}_{\perp}' &= \mathbf{A}_{\perp} \ , \end{aligned}$$

where $\mathbf{A} = \mathbf{A}_{\|} + \mathbf{A}_{\perp}$, with $\mathbf{A}_{\|}$ and $\mathbf{A}_{\perp}$ the components of $\mathbf{A}$ that are parallel and perpendicular to $\boldsymbol{\beta}$, respectively. Use the information given to show that the 3-vector $\mathbf{A}$ transforms under a Lorentz transformation according to

$$\mathbf{A}' = \mathbf{A} + \frac{\gamma - 1}{\beta^2} (\mathbf{A} \cdot \boldsymbol{\beta})\boldsymbol{\beta} - \gamma \boldsymbol{\beta} A_0 \ .$$

2. Use the general transformation equations for A_0 and $\mathbf{A}$ given in problem 1 to show explicitly that

$$A_0'^2 - \mathbf{A}'^2 = A_0^2 - \mathbf{A}^2 \ .$$

That is, the quantity $A_0^2 - \mathbf{A}^2$ is a *Lorentz scalar* - it has the same value in *all* inertial frames.

3. The scalar and vector potential together form the 4-vector potential, $(\Phi, \mathbf{A})$, where in inertial frame S,

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

and $\mathbf{B} = \nabla \times \mathbf{A}$. The Lorentz invariance of Maxwell's equations requires that in inertial frame S', the 4-vector potential is $(\Phi', \mathbf{A}')$ where the fields are

$$\mathbf{E}' = -\nabla'\Phi' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'}$$

and $\mathbf{B}' = \nabla' \times \mathbf{A}'$. Use these equations to determine the Lorentz transformations for the fields.