Mathematical Properties

In the special theory of relativity, we have seen that Lorentz transformations of the 4-vector $(ct, \vec{r})$ leave the quantity

$$s^2 = c^2t^2 - \vec{r}^{\,2} = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

invariant. The group of all transformations that leave s^2 invariant is called the homogeneous Lorentz group. It contains ordinary rotations as well as the Lorentz transformations already discussed.

The group of transformations that leave invariant

$$s^2(x, y) = (x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2$$

is called the inhomogeneous Lorentz group or the Poincaré group. It contains translations and reflections in both space and time, as well as the transformations of the homogeneous Lorentz group. We will restrict our discussion to the homogeneous transformations, and will henceforth omit the word homogeneous when referring to the Lorentz group.

It follows from the principle of relativity that the mathematical equations describing the laws of nature must be <u>covariant</u>, or invariant in form, under the transformations of the Lorentz group.
The space-time continuum is defined in terms of a four-dimensional space with coordinates,

$$x^\alpha = (x^0, x^1, x^2, x^3) = (ct, \vec{r}).$$

We assume that there is a well-defined transformation that yields new coordinates,

$$x'^\alpha = (x'^0, x'^1, x'^2, x'^3) = (ct', \vec{r}').$$

Tensors of rank $\underline{k}$ associated with $\underline{x}$ are defined by their transformation properties under the transformation $x \to x'$.
A <u>scalar</u> is a tensor of rank 0 — its value is not changed by the transformation.
A <u>vector</u> is a tensor of rank 1 — we distinguish two kinds:

$$A^\alpha = (A^0, \vec{A}) \qquad \underline{\text{contravariant vector}}$$

$$A_\alpha = (A^0, -\vec{A}) \qquad \underline{\text{covariant vector}}$$

The components A^0, A^1, A^2, A^3 of A^α transform according to the rule

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \qquad \underline{\text{Contravariant vector}}$$

where we use Einstein's summation convention: the repeated Lorentz index β implies a sum over $\beta = 0, 1, 2, 3$.

Similarly, the components A_0, A_1, A_2, A_3 of A_α transform according to the rule

$$A'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} A_\beta \qquad \underline{\text{covariant vector}}$$

For tensors of rank 2, we distinguish three kinds:

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} F^{\mu\nu} \qquad \underline{\text{contravariant tensor}}$$

$$F'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} F_{\mu\nu} \qquad \underline{\text{Covariant tensor}}$$

$$F'^{\alpha}{}_{\beta} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial x'^\beta} F^{\mu}{}_{\nu} \qquad \underline{\text{Mixed tensor}}$$

The generalization to tensors of arbitrary rank is obvious.

The <u>inner product</u> or scalar product of two vectors is defined as

$$\boxed{\begin{aligned} A \cdot B &\equiv A^{\alpha} B_{\alpha} = A_{\alpha} B^{\alpha} \\ &= A^0 B^0 - \vec{A} \cdot \vec{B} \end{aligned}}$$

The inner product is an invariant or scalar under the transformation:

$$\begin{aligned} A' \cdot B' = A'^{\alpha} B'_{\alpha} &= \left(\frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \right) \left(\frac{\partial x^{\gamma}}{\partial x'^{\alpha}} B_{\gamma} \right) \\ &= \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} A^{\beta} B_{\gamma} \\ &= \frac{\partial x^{\gamma}}{\partial x^{\beta}} A^{\beta} B_{\gamma} \\ &= \delta_{\beta}^{\gamma} A^{\beta} B_{\gamma} \\ &= A^{\beta} B_{\beta} \\ &= A \cdot B \end{aligned}$$

The invariant interval <u>ds</u> can be written generally as $\boxed{(ds)^2 = g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}}$, where $g_{\alpha\beta}$ is the <u>covariant metric tensor</u>. For the flat space-time of special relativity (in distinction to the curved space-time of general relativity), this tensor is diagonal, with elements

$$g_{00} = 1 \qquad g_{11} = g_{22} = g_{33} = -1 .$$

Similarly, we may write

$$(ds)^2 = g^{\alpha\beta} dx_\alpha \, dx_\beta = g^\alpha_\beta \, dx_\alpha \, dx^\beta .$$

In general, we have

$$\boxed{\begin{aligned} A^\alpha &= g^{\alpha\beta} A_\beta \\ A_\alpha &= g_{\alpha\beta} A^\beta \end{aligned}} .$$

Thus, $\quad A^\alpha = g^{\alpha\mu} A_\mu = g^{\alpha\mu} g_{\mu\beta} A^\beta$

$$\Rightarrow \quad \boxed{g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha_\beta}$$

where $\delta^\alpha_\beta = 0$ for $\alpha \neq \beta$ and $\delta^\alpha_\alpha = 1$ for $\alpha = 0,1,2,3$.

Consider now the partial derivative operators with respect to x^α and x_α. We have, for example,

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta}.$$

Thus, $\frac{\partial}{\partial x^\alpha}$ transforms like a covariant 4-vector:

$$\boxed{\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)}$$

Similarly, $\frac{\partial}{\partial x_\alpha}$ transforms like a contravariant 4-vector:

$$\frac{\partial}{\partial x'_\alpha} = \frac{\partial x_\beta}{\partial x'_\alpha}\frac{\partial}{\partial x_\beta}$$

(skip - this is not obvious consistent w/ contra. trf. law)

We write $\boxed{\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)}$

The 4-divergence of a 4-vector A is the invariant

$$\boxed{\begin{aligned}\partial\cdot A &= \partial^\alpha A_\alpha = \partial_\alpha A^\alpha \\ &= \frac{1}{c}\frac{\partial A^0}{\partial t} + \vec{\nabla}\cdot\vec{A}\end{aligned}}$$

The Laplacian operator is defined as

$$\boxed{\Box \equiv \partial^\alpha \partial_\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2}$$

Relativistic Momentum and Energy of a Particle

For a particle of mass m and velocity $|\vec{v}| \ll c$, the momentum is $\vec{p} = m\vec{v}$. The only generalization consistent with the Lorentz transformation law is given by the 4-vector,

$$p^\alpha = m U^\alpha = m(U^0, \vec{U}) = (p^0, \vec{p})$$

or,

$$p^\alpha = (m\gamma c, m\gamma\vec{v}).$$

Thus, the relativistic expression for the momentum of a particle with mass m and velocity $\vec{v}$ is

$$\boxed{\vec{p} = m\gamma\vec{v}}\ .$$

Now,*

$$p^\alpha p_\alpha = (mc)^2 = p_0^2 - \vec{p}^{\,2}$$

$$\Rightarrow \quad p_0 = \sqrt{(mc)^2 + \vec{p}^{\,2}}$$

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* $p^0 = p_0$

Equivalently, we have

$$P_0 = m\gamma c = \frac{mc}{\sqrt{1-v^2/c^2}}$$

For $v \ll c$, $\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \simeq 1 + \frac{1}{2}\frac{v^2}{c^2}$.

$$\Rightarrow \quad P_0 \simeq mc\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) \quad ; \; v \ll c$$

or,

$$\boxed{P_0 c \simeq mc^2 + \frac{1}{2}mv^2}$$

Thus, $P_0 c - mc^2$ is the relativistic generalization of the particle's kinetic energy. We identify the total energy of the particle as the quantity,

$$\boxed{E = P_0 c = mc^2\gamma},$$

so the kinetic energy is

$$\boxed{T = mc^2(\gamma - 1)}.$$

$$\left\{ \text{Note: } \frac{pc}{E} = \beta \right\}$$

We can write

$$\boxed{E = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \sqrt{(mc^2)^2 + (\vec{p}c)^2}}$$

and $p^\alpha = \left(\frac{E}{c}, \vec{p}\right)$.

The relativistic generalization of Newton's second law of motion is

$$\boxed{F^\alpha = \frac{dp^\alpha}{d\tau}} \; , \qquad \left(\begin{array}{l} F^\alpha \text{ is called the} \\ \text{Minkowski force} \end{array}\right)$$

where $\underline{F}^\alpha$ is a force 4-vector. Now,

$$F^\alpha = \frac{dp^\alpha}{dt}\frac{dt}{d\tau} = \gamma\frac{dp^\alpha}{dt}$$

$$\Rightarrow \quad F^\alpha = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right) \equiv \gamma\left(\frac{1}{c}\frac{dE}{dt}, \vec{F}\right)$$

Consider the invariant quantity, $F^\alpha dX_\alpha$. We have

$$F \cdot dx = \gamma\left(\frac{1}{c}\frac{dE}{dt}\right)(c\,dt) - \gamma\frac{d\vec{p}}{dt}\cdot d\vec{r}$$

$$= \gamma\left(dE - \vec{F}\cdot d\vec{r}\right)$$

But $\quad F\cdot dx = \frac{dP}{d\tau}\cdot dx = U\cdot dP$

For a particle of mass $\underline{m}$, we have

$$F\cdot dx = \frac{1}{m}P\cdot dP = \frac{1}{2m}d(P\cdot P) = 0$$

$$\boxed{F\cdot dx = 0} \Rightarrow \boxed{dE = \vec{F}\cdot d\vec{r}} \; ,$$

which is the <u>work-energy theorem</u>.

Gaussian Units

$$\left\{\begin{array}{l} \vec{\nabla}\cdot\vec{D} = 4\pi\rho \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \end{array}\right\}$$

[Maxwell's Equations]

$$\left\{\begin{array}{l} \vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \end{array}\right\}$$

[E-m Potentials]

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$$

(Lorentz Condition)

$$\vec{F} = q\left(\vec{E} + \frac{\vec{V}}{c}\times\vec{B}\right)$$

[Lorentz Force]

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

(Charge Continuity Equation)

Note that we can write

$$F^{\alpha} = m \frac{dU^{\alpha}}{d\tau}$$

for a particle of mass $\underline{m}$, where $\frac{dU^{\alpha}}{d\tau}$ is an acceleration 4-vector.

Covariance of Electrodynamics

Consider the microscopic Maxwell equations. The charge density ρ and current density $\underline{\vec{J}}$ satisfy the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \, .$$

It is natural to postulate that ρ and $\vec{J}$ together form a 4-vector $\underline{J^{\alpha}}$:

$$\boxed{J^{\alpha} = (c\rho, \vec{J})} \, .$$

Then the continuity equation takes the manifestly covariant form

$$\boxed{\partial \cdot J = 0} \, .$$

In the Lorentz gauge,

$$\frac{1}{c} \frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0 \, .$$

If we identify Φ and $\vec{A}$ as components of the 4-vector potential A^α,

$$\boxed{A^\alpha = (\Phi, \vec{A})},$$

then the Lorentz condition can be written as

$$\boxed{\partial \cdot A = 0}.$$

With this choice of gauge, the potentials Φ and $\vec{A}$ satisfy inhomogeneous wave equations:

$$\left\{ \begin{aligned} &\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \vec{\nabla}^2 \Phi = 4\pi\rho \\ &\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} = \frac{4\pi}{c}\vec{J} \end{aligned} \right.$$

or,

$$\boxed{\Box A^\alpha = \frac{4\pi}{c} J^\alpha}.$$

The fields $\vec{E}$ and $\vec{B}$ are given by

$$\left\{ \begin{aligned} &\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ &\vec{B} = \vec{\nabla} \times \vec{A} \end{aligned} \right.$$

Thus,

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial x^0} = -\frac{\partial A^0}{\partial x} - \frac{\partial A^1}{\partial x^0}$$

or,

$$E_x = -(\partial^0 A^1 - \partial^1 A^0)$$

Similarly,
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$
$$= \frac{\partial A^3}{\partial y} - \frac{\partial A^2}{\partial z}$$
$$= -\left(\partial^2 A^3 - \partial^3 A^2\right)$$

These equations suggest that the six components of $\vec{E}$ and $\vec{B}$ are the elements of a <u>second-rank, asymmetric field-strength tensor</u>:

$$\boxed{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha},$$

where $F^{\alpha\beta} = -F^{\beta\alpha}$. In matrix form,

$$\boxed{F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}}.$$

The inhomogeneous Maxwell equations,

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\[1em] \vec{\nabla} \times \vec{B} - \dfrac{1}{c}\dfrac{\partial \vec{E}}{\partial t} = \dfrac{4\pi}{c}\vec{J} \end{array} \right.$$