can be written in manifestly covariant form as

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta} \, .$$

Similarly, the homogeneous Maxwell equations,

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \end{cases}$$

can be written as the four equations,

$$\boxed{\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0} \, .$$

We may show that (see next page)

$$\boxed{\Box F^{\alpha\beta} = +\frac{4\pi}{c} \left(\partial^\alpha J^\beta - \partial^\beta J^\alpha \right)} \, .$$

To complete the discussion, for a particle of charge q in an electromagnetic field, the Lorentz force and rate of change of energy are given by

$$\begin{cases} \dfrac{d\vec{p}}{dt} = q\left(\vec{E} + \dfrac{\vec{v}}{c} \times \vec{B} \right) = \vec{F} \\[2ex] \dfrac{dE}{dt} = q \, \vec{E} \cdot \vec{v} = \vec{F} \cdot \dfrac{d\vec{r}}{dt} = \left(\dfrac{dP_0}{dt} \right) c \end{cases}$$

The 4-vector force is then

$$F^\alpha = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \vec{F}\right) = \frac{dp^\alpha}{d\tau}$$

and, in general, we can write

$$\boxed{\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta}\ ,$$

where $U^\beta = (\gamma c, +\gamma\vec{v})$ is the 4-vector velocity.

Aside: We have $\partial_\gamma F^{\gamma\beta} = \frac{4\pi}{c} J^\beta$

$$\Rightarrow \partial^\alpha \partial_\gamma F^{\gamma\beta} = \frac{4\pi}{c}\partial^\alpha J^\beta$$

$$\text{and,}\quad \partial^\beta \partial_\gamma F^{\gamma\alpha} = \frac{4\pi}{c}\partial^\beta J^\alpha$$

$$\therefore\ \partial_\gamma\left(\partial^\alpha F^{\gamma\beta} - \partial^\beta F^{\gamma\alpha}\right) = \frac{4\pi}{c}\left(\partial^\alpha J^\beta - \partial^\beta J^\alpha\right)$$

$$\text{or,}\quad \partial_\gamma\left(\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha}\right) = -\frac{4\pi}{c}\left(\partial^\alpha J^\beta - \partial^\beta J^\alpha\right)$$

$$\text{But}\quad \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} = -\partial^\gamma F^{\alpha\beta}$$

$$\Rightarrow \partial_\gamma \partial^\gamma F^{\alpha\beta} = \frac{4\pi}{c}\left(\partial^\alpha J^\beta - \partial^\beta J^\alpha\right)$$

$$\text{or,}\quad \boxed{\Box F^{\alpha\beta} = \frac{4\pi}{c}\left(\partial^\alpha J^\beta - \partial^\beta J^\alpha\right)}$$

For the macroscopic equations,

$$\begin{cases} \vec{\nabla}\cdot\vec{D} = 4\pi\rho_f \\ \vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}_f \end{cases},$$

we have $$\boxed{\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} J^\beta_f},$$

where, in matrix form,

$$\boxed{G^{\alpha\beta} = \begin{bmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{bmatrix}}.$$

$G^{\alpha\beta}$ is called the <u>macroscopic field-strength tensor</u>. We can write

$$\boxed{G^{\alpha\beta} \equiv F^{\alpha\beta} + 4\pi Q^{\alpha\beta}}$$

where

$$Q^{\alpha\beta} = \begin{bmatrix} 0 & -P_x & -P_y & -P_z \\ P_x & 0 & M_z & -M_y \\ P_y & -M_z & 0 & M_x \\ P_z & M_y & -M_x & 0 \end{bmatrix}$$

with $\vec{P}$ the polarization and $\vec{M}$ the magnetization.

With $\quad J^\beta = J_f^\beta + J_b^\beta$,

we have $\quad \partial_\alpha F^{\alpha\beta} + 4\pi\,\partial_\alpha Q^{\alpha\beta} = \frac{4\pi}{c} J_f^\beta$

or, $\quad \frac{4\pi}{c}(J_f^\beta + J_b^\beta) + 4\pi\,\partial_\alpha Q^{\alpha\beta} = \frac{4\pi}{c} J_f^\beta$

$$\therefore \quad \boxed{J_b^\beta = -c\,\partial_\alpha Q^{\alpha\beta}}$$

Transformation of Electromagnetic Fields

The tensor $F^{\alpha\beta}$ transforms according to the rule

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} F^{\mu\nu} .$$

Similarly, the 4-vector A^α transforms according to the rule

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\mu} A^\mu .$$

Consider a general Lorentz transformation from $\underline{S}$ to $\underline{S'}$ moving with velocity $\vec{v} = \vec{\beta} c$ relative to $\underline{S}$. For this case, we have seen (see pages 11-17 and 11-18):

$$A_0' = \gamma(A_0 - \vec{\beta}\cdot\vec{A})$$

$$\vec{A}' = \vec{A} + \frac{(\gamma-1)}{\beta^2}(\vec{A}\cdot\vec{\beta})\vec{\beta} - \gamma\vec{\beta}A_0$$

We can write these equations in matrix form as

$$\begin{bmatrix} A_0' \\ A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_y\beta_x}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_z\beta_x}{\beta^2} & \frac{(\gamma-1)\beta_z\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2} \end{bmatrix} \begin{bmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{bmatrix}$$

Hence, we can write $\underset{\sim}{A}' = \underset{\sim}{\mathcal{L}}\, \underset{\sim}{A}$,

where $\underset{\sim}{\mathcal{L}}$ is the 4×4 Lorentz transformation matrix and $\underset{\sim}{A}$ is a 4×1 column matrix. The corresponding transformation for a second-rank tensor is

$$\underset{\sim}{F}' = \underset{\sim}{\mathcal{L}}\, \underset{\sim}{F}\, \underset{\sim}{\mathcal{L}}^T ,$$

where $\underset{\sim}{F}$ is a 4×4 matrix and $\underset{\sim}{\mathcal{L}}^T$ is the transpose of $\mathcal{L}$. These equations lead to the transformation of the fields: *

$$\boxed{\begin{aligned} \vec{E}' &= \gamma\left(\vec{E} + \vec{\beta} \times \vec{B}\right) - \frac{\gamma^2}{\gamma+1}\, \vec{\beta}\,(\vec{\beta} \cdot \vec{E}) \\ \vec{B}' &= \gamma\left(\vec{B} - \vec{\beta} \times \vec{E}\right) - \frac{\gamma^2}{\gamma+1}\, \vec{\beta}\,(\vec{\beta} \cdot \vec{B}) \end{aligned}}$$

(See page 11–46–1)

These equations show that a purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another. Thus, $\vec{E}$ and $\vec{B}$ have no independent existence.

Note that if $v \ll c$, so that terms of order β^2 can be neglected, we have

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* See note at bottom of next page.

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} F^{\mu\nu}.$$

Let $\mathcal{L}_{\alpha\mu} \equiv \dfrac{\partial x'^{\alpha}}{\partial x^{\mu}}$.

Then $\mathcal{L}_{\beta\nu} = \dfrac{\partial x'^{\beta}}{\partial x^{\nu}}$.

$$\Rightarrow F'^{\alpha\beta} = \mathcal{L}_{\alpha\mu} F^{\mu\nu} \mathcal{L}_{\beta\nu}$$

$$= \mathcal{L}_{\alpha\mu} F^{\mu\nu} (\underset{\sim}{\mathcal{L}}^{T})_{\nu\beta}$$

$$\therefore \underset{\sim}{F}' = \underset{\sim}{\mathcal{L}}\, \underset{\sim}{F}\, \underset{\sim}{\mathcal{L}}^{T}.$$

$$\begin{cases} \vec{E}' \simeq \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \\ \vec{B}' \simeq \vec{B} - \frac{\vec{v}}{c} \times \vec{E} \end{cases}$$

Similarly, for the polarization $\vec{P}$ and magnetization $\vec{M}$, we have

$$\begin{cases} \vec{P}' \simeq \vec{P} - \frac{\vec{v}}{c} \times \vec{M} \\ \vec{M}' \simeq \vec{M} + \frac{\vec{v}}{c} \times \vec{P} \end{cases}$$

with the inverse transformations,

$$\begin{cases} \vec{P} \simeq \vec{P}' - \left(\vec{M}' \times \frac{\vec{v}}{c}\right) \\ \vec{M} \simeq \vec{M}' + \left(\vec{P}' \times \frac{\vec{v}}{c}\right) \end{cases}$$

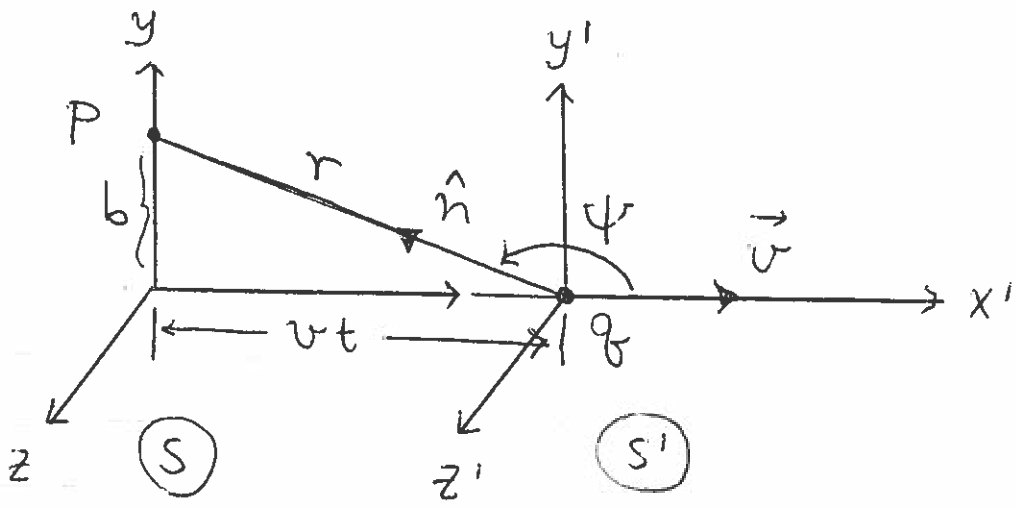
As an example of the transformation of the electromagnetic fields, consider a charge q at rest in frame S'. The charge as seen by an observer in frame S moves with constant velocity $\vec{v} = v\,\hat{x}$.

Note that the transformation equations for fields imply that

$$\vec{E}' \cdot \vec{B}' = \vec{E} \cdot \vec{B} \quad \text{and} \quad \vec{E}'^2 - \vec{B}'^2 = E^2 - B^2 .$$

$$\begin{cases} \vec{E}'_{\parallel} = \vec{E}_{\parallel} \\ \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}_{\perp}) \end{cases}$$

$$\begin{cases} \vec{B}'_{\parallel} = \vec{B}_{\parallel} \\ \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \vec{\beta} \times \vec{E}_{\perp}) \end{cases}$$



In frame S', where q is at rest, the fields are

$$\begin{cases} \vec{E}' = \dfrac{q\vec{r}'}{r'^3} = \dfrac{q}{r'^3}(x'\hat{x} + y'\hat{y} + z'\hat{z}) \\ \vec{B}' = 0 \end{cases}$$

where, at the observer's point P, we have

$$\begin{cases} x' = -vt' \\ y' = b \\ z' = 0 \end{cases}$$

We can write $\vec{E}' = E_x'\hat{x} + E_y'\hat{y} + E_z'\hat{z}$, where

$$\begin{cases} E_x' = \dfrac{-qvt'}{(b^2 + v^2t'^2)^{3/2}} \\ E_y' = \dfrac{qb}{(b^2 + v^2t'^2)^{3/2}} \end{cases}$$

and $E'_z = 0$. We can write

$$t' = \gamma\left(t - \frac{vx}{c^2}\right),$$

where at the observer's point P, we have

$$\begin{cases} x = 0 \\ y = b \\ z = 0 \end{cases}$$

Thus, $t' = \gamma t$, and we find that

$$\boxed{\begin{aligned} E_x' &= \frac{-q\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \\ E_y' &= \frac{qb}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \\ E_z' &= 0 \end{aligned}}$$

Electric field in frame S′ expressed in terms of coordinates of frame S.

Since $\vec{B}' = 0$, the fields in frame $\underline{S}$ are

$$\begin{cases} \vec{E} = \gamma \vec{E}' - \dfrac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}') \\ \vec{B} = \gamma\vec{\beta}\times\vec{E}' = \dfrac{\vec{v}}{c}\times\vec{E} \end{cases}$$

where $\vec{\beta} = \dfrac{\vec{v}}{c} = \dfrac{v}{c}\hat{x} = \beta\hat{x}$.

Thus,

$$\begin{cases} E_x = \gamma E_x' - \dfrac{\gamma^2}{\gamma+1}\beta(\beta E_x') \\ E_y = \gamma E_y' \\ E_z = \gamma E_z' = 0 \end{cases}$$

and $\vec{B} = \gamma\beta E_y' \hat{z}$.

Now $\gamma^2 = \dfrac{1}{1-\beta^2} \Rightarrow \gamma^2 - \gamma^2\beta^2 = 1$

$$\therefore \frac{\gamma^2\beta^2}{\gamma+1} = \frac{\gamma^2-1}{\gamma+1} = \gamma - 1$$

$$\Rightarrow \quad E_x = \gamma E_x' - (\gamma-1)E_x' = E_x'$$

Hence, we find

$$\boxed{\begin{aligned} E_x &= \frac{-q\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \\ E_y &= \frac{\gamma q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \\ E_z &= 0 \end{aligned}}$$

and

$$\boxed{\begin{aligned} B_x &= 0 \\ B_y &= 0 \\ B_z &= \frac{\gamma q v b}{c\,(b^2+\gamma^2 v^2 t^2)^{3/2}} \end{aligned}}$$

These expressions for the fields emphasize their time dependence at a fixed observation point, An alternative description can be given in terms of the spatial variation of the fields relative to the instantaneous <u>present position</u> of the charge in the lab.

Let $\quad \hat{n}\cdot\hat{v} \equiv \cos\psi \quad$ [see figure on page 11-47].

Then

$$\begin{aligned} \vec{r} &= r\,\hat{n} \\ &= r\,(\hat{x}\cos\psi + \hat{y}\sin\psi) \\ &= -vt\,\hat{x} + b\,\hat{y} \end{aligned}$$

Now

$$\begin{aligned} b^2+\gamma^2 v^2 t^2 &= r^2\sin^2\psi + \gamma^2 r^2\cos^2\psi \\ &= r^2\sin^2\psi + \gamma^2 r^2(1-\sin^2\psi) \\ &= \gamma^2 r^2 - r^2(\gamma^2-1)\sin^2\psi \\ &= \gamma^2 r^2 - \gamma^2\beta^2 r^2\sin^2\psi \end{aligned}$$

$$b^2 + \gamma^2 v^2 t^2 = r^2 \gamma^2 (1 - \beta^2 \sin^2 \psi)$$

$$(b^2 + \gamma^2 v^2 t^2)^{3/2} = r^3 \gamma^3 (1 - \beta^2 \sin^2 \psi)^{3/2}$$

Thus,

$$\left\{ \begin{aligned} E_x &= \frac{-q v t}{r^3 \gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}} \\ E_y &= \frac{q b}{r^3 \gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}} \\ E_z &= 0 \end{aligned} \right.$$

or,

$$\boxed{\vec{E} = \frac{q \vec{r}}{r^3 \gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}}}$$

Thus, the electric field is radial, but the field lines are distributed isotropically only for $\beta = 0$.

Since $\vec{v} \times \vec{r} = v b \hat{z}$, the magnetic field can be written as

$$\boxed{\vec{B} = \frac{q}{c} \frac{\vec{v} \times \vec{r}}{r^3 \gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}}}.$$

$$\Rightarrow \quad \vec{B} = \frac{\vec{v}}{c} \times \vec{E} = \vec{\beta} \times \vec{E}$$

(cf. page 11-48)

At nonrelativistic speeds, we fin

$$\vec{B} \simeq \frac{q}{c} \frac{\vec{v} \times \vec{r}}{r^3},$$

which is just the Ampère-Biot-Sa expression for the magnetic fiel moving charge.