Poincaré Invariant Three-Body Scattering at Intermediate Energies

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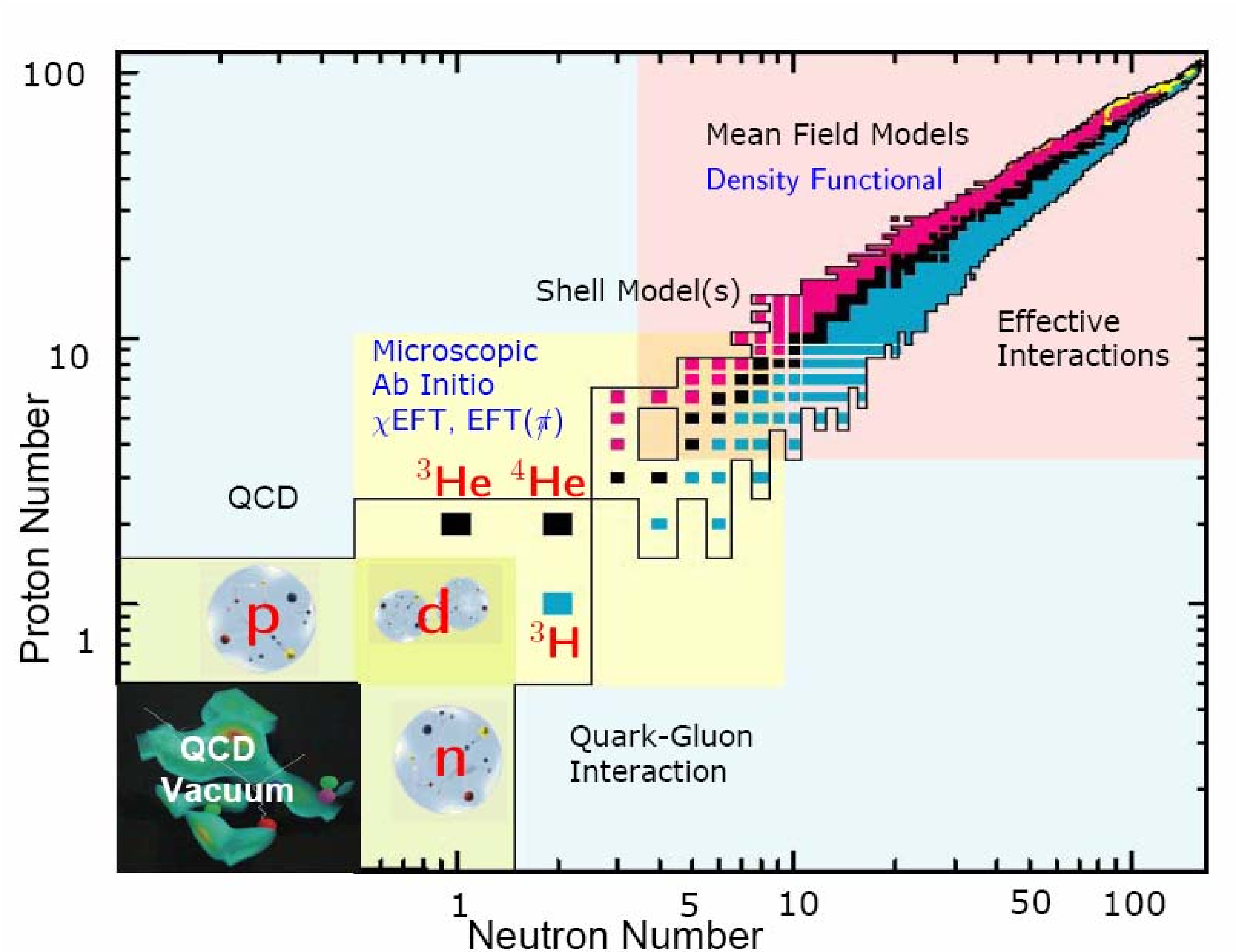
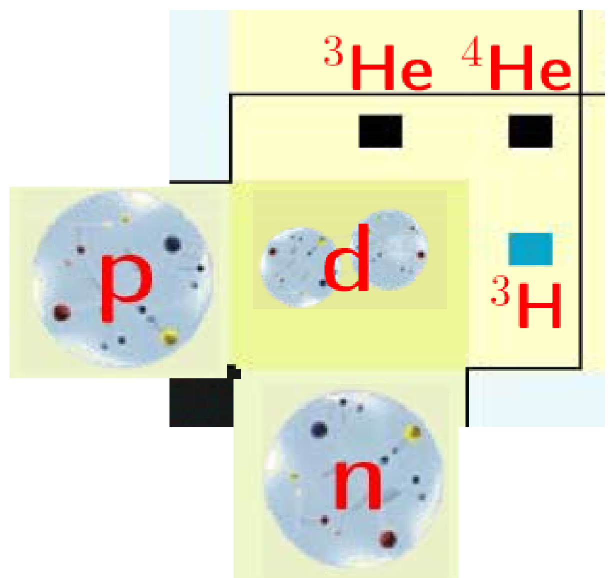
Supported by: U.S. DOE, OSC, NERSC

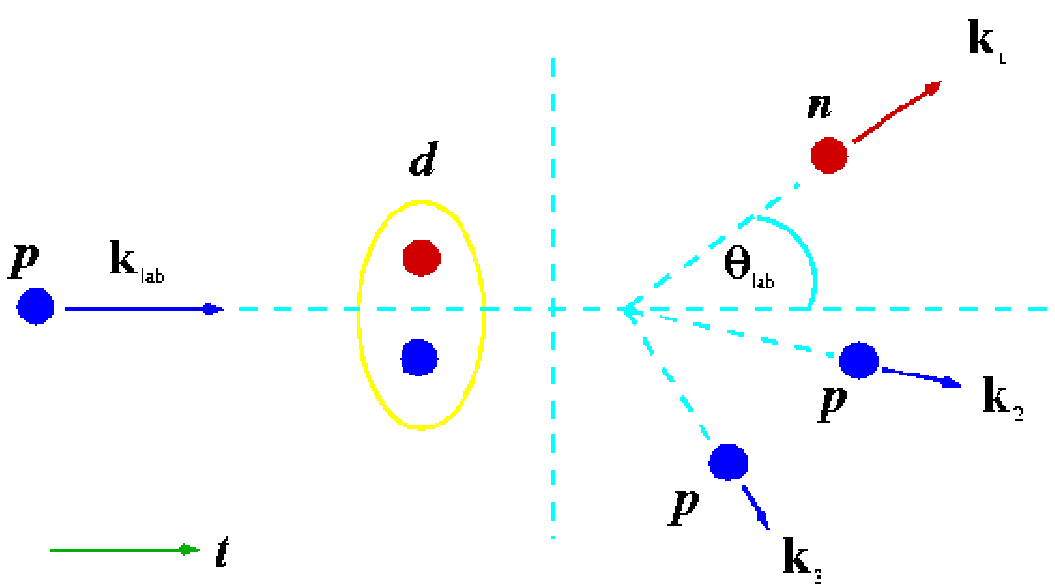

Mean Field Models
Density Functional
Shell Model(s)
Effective
Interactions
Microscopic
Ab Initio
χEFT, EFT(π̸)
QCD
^{3}He
^{4}He
p
d
^{3}H
n
QCD
Vacuum
Quark-Gluon
Interaction
Proton Number
Neutron Number
100
10
1
1
5
10
50
100

A Few-Body Theorist's view of the Nuclear Chart



3 Nucleon Systems

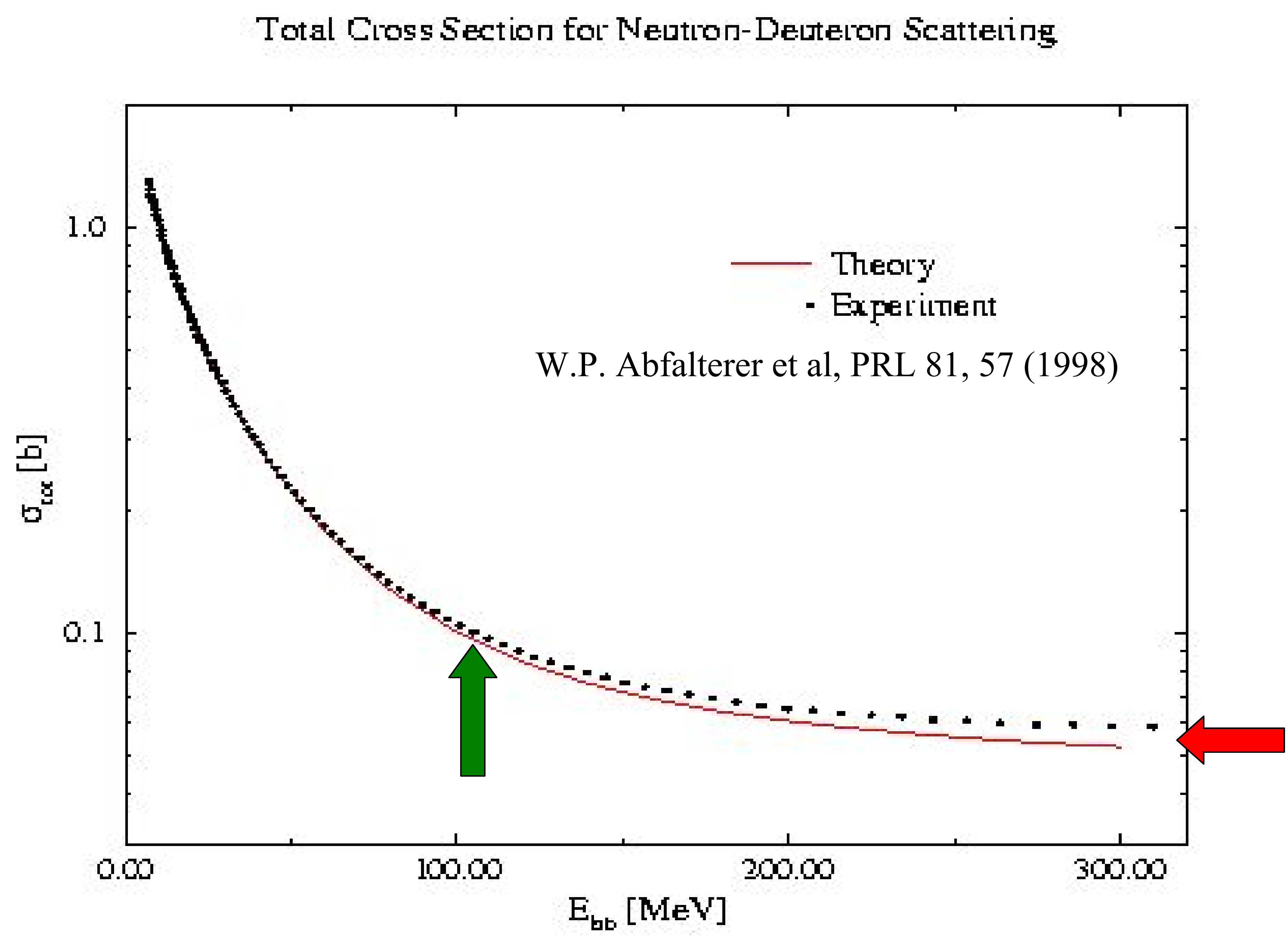
- Bound State: ^{3}H - ^{3}He
- Scattering: **Elastic** – **Inelastic (Breakup)**
- Energy Scale: keV → MeV → GeV



Challenges in 3N Physics

- **Test of nuclear forces in the simplest nuclear environment** (over a large energy range!)
  - Two-body forces
  - Genuine three-body forces

- **Reaction mechanisms**
  - Examples: deuteron breakup, (p,n) charge exchange, exclusive breakup (specific configurations) …
  - Higher Energy: Lorentz vs. Galilean Invariance
  - Check commonly used approximations (e.g. Glauber approach)



Total Cross Section for Neutron-Deuteron Scattering
1.0
0.1
Theory
Experiment
W.P. Abfalterer et al, PRL 81, 57 (1998)
σtot [b]
0.00
100.00
200.00
300.00
Elab [MeV]

Relativistic Effects at Higher Energies

Computational Challenge:

3N and 4N systems:

- standard treatment based on pw projected momentum space successful (3N scattering up to ≈250 MeV) but rather tedious
- 2N: j_{max}=5, 3N: J_{max}=25/2 → 200 `channels'
- Computational maximum today:
- 2N: j_{max}=7, 3N: J_{max}=31/2

⇒ **Solution:**

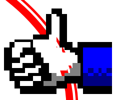
⇒ **NO partial wave decomposition of basis states**

Roadmap for 3N problem without PW

Scalar NN model | Realistic NN Model

- NN scattering + bound state
- 3N bound state
- 3N bound state + 3NF
- 3N scattering:
- Full Faddeev Calculation
  - Elastic scattering
  - Below and above break-up
  - Break-up

- **Poincarė Invariant Faddeev Calculations**

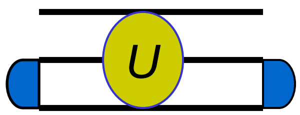


- NN scattering + deuteron
  - Potentials AV18 and Bonn-B
- Break-up in first order:
  - (p,n) charge exchange
  - Max. Energy 500 MeV
  - Lorentz kinematics

- Exact Faddeev Calculation
  - NN interactions
  - High energy limits

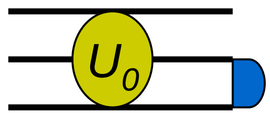
Three-Body Scattering - General

- Transition operator for elastic scattering



$$U = PG_0^{-1} + PT$$

- Transition operator for breakup scattering



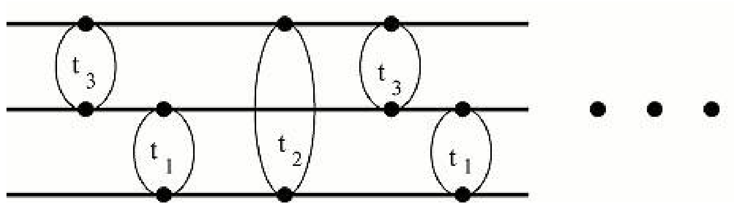
$$U_0 = (1 + P)T$$

$$T = tP + tG_0 PT$$

- Faddeev equation (Multiple Scattering Series)

$$T = tP + tG_0 PtP + \cdots$$

1st Order in tP



$t = v + vg_0 t$ =: NN t-matrix

$P = P_{12} P_{23} + P_{13} P_{23} \equiv$ Permutation Operator

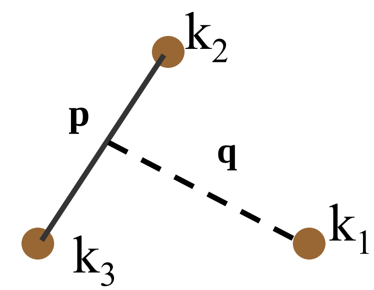
3-Body Transition Amplitude (NR)

$$T\left|\mathrm{q}_0\varphi_d\right\rangle = tP\left|\mathrm{q}_0\varphi_d\right\rangle + tG_0PT\left|\mathrm{q}_0\varphi_d\right\rangle$$

$$\mathrm{p} = \frac{1}{2}\left(\mathrm{k}_2 - \mathrm{k}_3\right)$$

$$\mathrm{q} = \frac{2}{3}\left(\mathrm{k}_1 - \frac{1}{2}(\mathrm{k}_2 + \mathrm{k}_3)\right)$$

The Faddeev Equation in momentum space by using Jacobi Variables

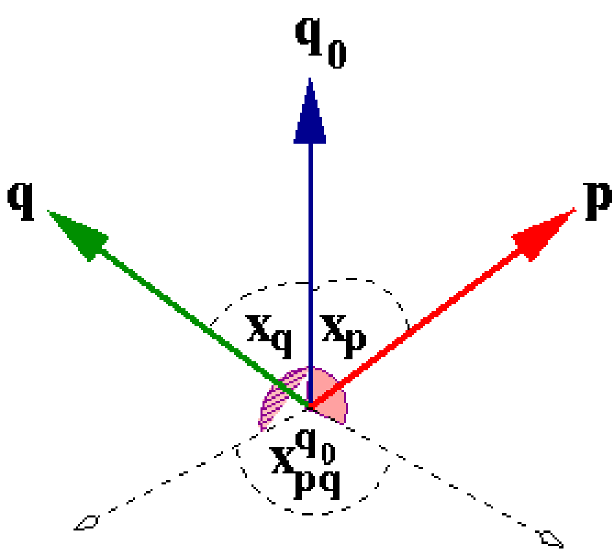


$$\left\langle \mathrm{pq}\left|\hat{T}\right|\mathrm{q}_0\varphi_d\right\rangle = \varphi_d\left(\mathrm{q}+\tfrac{1}{2}\mathrm{q}_0\right)\hat{t}_s\left(\mathrm{p},\tfrac{1}{2}\mathrm{q}+\mathrm{q}_0, E-\tfrac{3}{4m}q^2\right)$$
$$+\int d^3\mathrm{q}'' \frac{\hat{t}_s\left(\mathrm{p},\tfrac{1}{2}\mathrm{q}+\mathrm{q}'', E-\tfrac{3}{4m}q^2\right)}{E-\tfrac{1}{m}\left(q^2+q''^2+\mathrm{q}\cdot\mathrm{q}''\right)+i\varepsilon}\,\frac{\left\langle \mathrm{q}+\tfrac{1}{2}\mathrm{q}'',\mathrm{q}''\left|\hat{T}\right|\mathrm{q}_0\varphi_d\right\rangle}{E-\tfrac{3}{4m}q''^2-E_d+i\varepsilon}$$

$\hat{t}_s \equiv$ symmetrized 2-body t-matrix

Variables for 3D Calculation

3 distinct vectors in the problem: $\mathbf{q}_0$ $\mathbf{q}$ $\mathbf{p}$



5 independent variables:

$$p = |\mathrm{p}| \ , \ q = |\mathrm{q}|$$

$$x_p = \hat{\mathrm{p}} \cdot \hat{\mathrm{q}}_0 \ , x_q = \hat{\mathrm{q}} \cdot \hat{\mathrm{q}}_0$$

$$x_{pq}^{q_0} = (\mathrm{q}_0 \times \mathrm{q}) \cdot (\mathrm{q}_0 \times \mathrm{p})$$

q system : **z || q**

q_0 system : **z || $\mathbf{q_0}$**

Variables invariant under rotation:

freedom to choose coordinate system for numerical calculation

Relativistic Faddeev Calculations

- Context: Poincarė Invariant Quantum Mechanics
  - Poincarė invariance is exact symmetry, realized by a unitary representation of the Poincarė group on a few-particle Hilbert space
  - Instant form
  - Faddeev equations same operator form but different ingredients
- **Kinematics**
  - Lorentz transformations between frames
- **Dynamics**
  - Bakamjian-Thomas Scheme: Mass Operator $M=M_0+V$ replaces Hamiltonian $H=H_0+\nu$
  - Connect Galilean two-body ν with Poincarė two-body v
  - Construct V := $\sqrt{M^2+q^2}-\sqrt{M_0^2+q^2}$

Lorentz Kinematics: Phase Space Factors

$$\sigma_{el} = (2\pi)^4 \int d\Omega \, \frac{E_n^2(q_0) E_d^2(q_0)}{W} \left| \langle \varphi_d \hat{\mathrm{q}} q_0 | U | \varphi_d \mathrm{q}_0 \rangle \right|^2$$

NR: $(2m/3)^2$

$$W = \sqrt{4(m^2+p^2)+q^2} + \sqrt{m^2+q^2} \quad \equiv \text{Invariant Mass}$$

$$\sigma_{br} = \frac{(2\pi)^4}{3} \frac{E_n(q_0)E_d(q_0)}{q_0 W} \int d\Omega_p d\Omega_q dq \, \frac{p_u q^2}{4} \sqrt{4(m^2+p_u^2)+q^2} \left| \langle \phi_0 | U_0 | \varphi_d \mathrm{q}_0 \rangle \right|^2$$

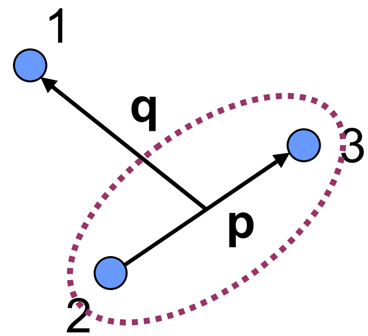
$$|\mathrm{p_u}| = 1/2 \sqrt{W^2 - 3m^2 - 2W\sqrt{m^2+q^2}}$$

$$\sigma_{br}^{NR} = \frac{(2\pi)^4}{3} \frac{m^2}{3q_0} \int d\Omega_p d\Omega_q dq q^2 \sqrt{mE_{cm} - \tfrac{3}{4} q^2} \left| \langle \phi_0 | U_0 | \varphi_d \mathrm{q}_0 \rangle \right|^2$$

Kinematics: Poincaré-Jacobi momenta

- **Nonrelativistic (Galilei)**

$$
\begin{aligned}
\mathrm{p} &= \frac{1}{2}(\mathrm{k}_2 - \mathrm{k}_3) \\
\mathrm{q} &= \frac{2}{3}(\mathrm{k}_1 - \frac{1}{2}(\mathrm{k}_2 + \mathrm{k}_3))
\end{aligned}
$$



- Relativistic (Lorentz)

$$
\begin{aligned}
\mathrm{p} &= \frac{1}{2}(\mathrm{k}_2 - \mathrm{k}_3) + \frac{\mathrm{k}_2 + \mathrm{k}_3}{2m_{23}} \left(\frac{(\mathrm{k}_2 - \mathrm{k}_3)\cdot(\mathrm{k}_2 + \mathrm{k}_3)}{(E_2 + E_3) + m_{23}} - (E_2 - E_3) \right) \\
\mathrm{q} &= \mathrm{k}_1 + \frac{\mathbf{K}}{M} \left(\frac{\mathrm{k}_1 \cdot \mathbf{K}}{E + M} - E_1 \right)
\end{aligned}
$$

$$
\begin{aligned}
E &= E_1 + E_2 + E_3 \\
\mathbf{K} &= \mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3 \\
M &= \sqrt{E^2 - \mathbf{K}^2} \\
m_{23} &= \sqrt{(E_2 + E_3)^2 - (\mathrm{k}_2 + \mathrm{k}_3)^2}
\end{aligned}
$$

Kinematics: Poincaré-Jacobi Coordinates

3N c.m. frame: k_1, k_2, k_3 with $k_1 + k_2 + k_3 = K = 0$

Poincarė-Jacobi Coordinates:

$$\mathrm{q} = \mathrm{k}_1$$

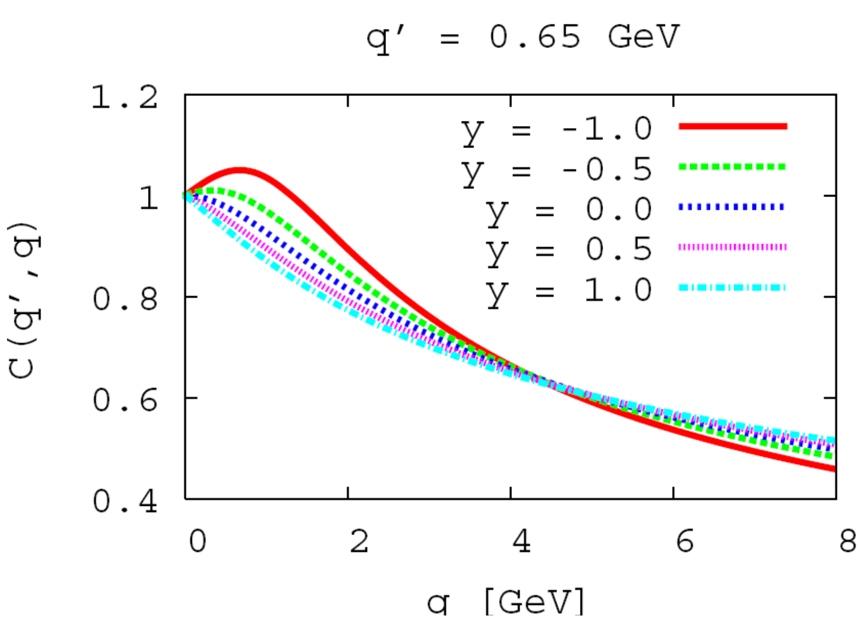
$$\mathrm{p} = \frac{1}{2}(\mathrm{k}_2 - \mathrm{k}_3) - \frac{1}{2}(\mathrm{k}_2 + \mathrm{k}_3)\left(\frac{E_2 - E_3}{E_2 + E_3 + \sqrt{(E_2 + E_3)^2 - (\mathrm{k}_2 + \mathrm{k}_3)^2}}\right)$$

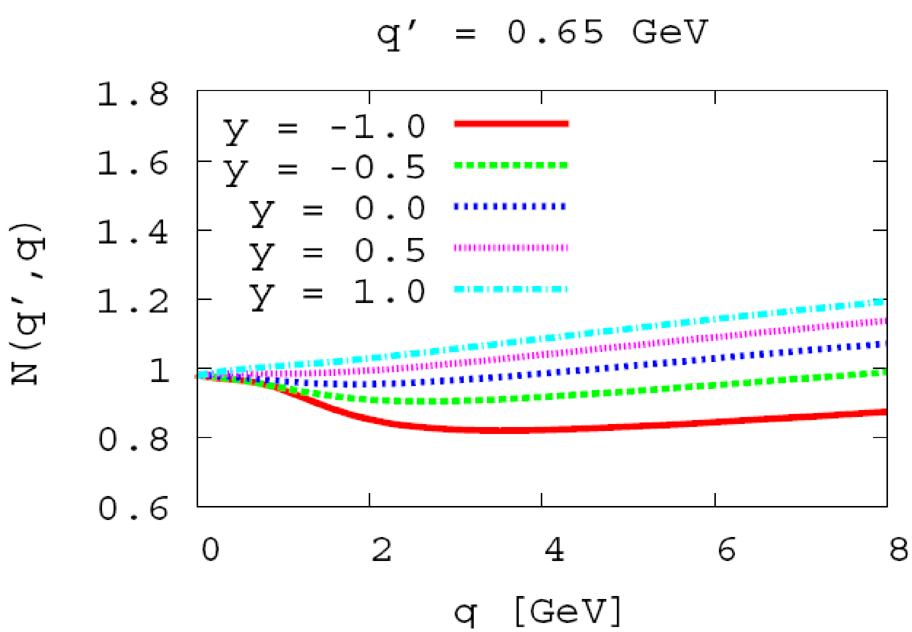
$$|\mathrm{k}_1\mathrm{k}_2\mathrm{k}_3\rangle = \left|\frac{\partial(K\mathrm{pq})}{\partial(\mathrm{k}_2\mathrm{k}_3)}\right|^{1/2} |\mathrm{Kpq}\rangle = \frac{E(p)[E(k_2) + E(k_3)]}{2E(k_2)E(k_3)} |\mathrm{Kpq}\rangle$$

- All expressions related to permutations much more complicated
- Depend on vector variables => angle dependent

Permutation Operator: $P=P_{12}P_{23}+P_{13}P_{23}$

$$
\begin{aligned}
{}_1\langle \mathbf{p}'\mathbf{q}'|P|\mathbf{p}''\mathbf{q}''\rangle_1 &= {}_1\langle \mathbf{p}'\mathbf{q}'|\mathbf{p}''\mathbf{q}''\rangle_2 + {}_1\langle \mathbf{p}'\mathbf{q}'|\mathbf{p}''\mathbf{q}''\rangle_3 \\
&= \hat{N}(\mathbf{q}',\mathbf{q}'')\Big[\delta\left(\mathbf{p}'-\mathbf{q}''-\frac{1}{2}\mathbf{q}'\underline{C(\mathbf{q}'',\mathbf{q}')}\right)\delta\left(\mathbf{p}''+\mathbf{q}'+\frac{1}{2}\mathbf{q}''\underline{C(\mathbf{q}',\mathbf{q}'')}\right) \\
&\quad +\delta\left(\mathbf{p}'+\mathbf{q}''+\frac{1}{2}\mathbf{q}'\underline{C(\mathbf{q}'',\mathbf{q}')}\right)\delta\left(\mathbf{p}''-\mathbf{q}'-\frac{1}{2}\mathbf{q}''\underline{C(\mathbf{q}',\mathbf{q}'')}\right)\Big]
\end{aligned}
$$





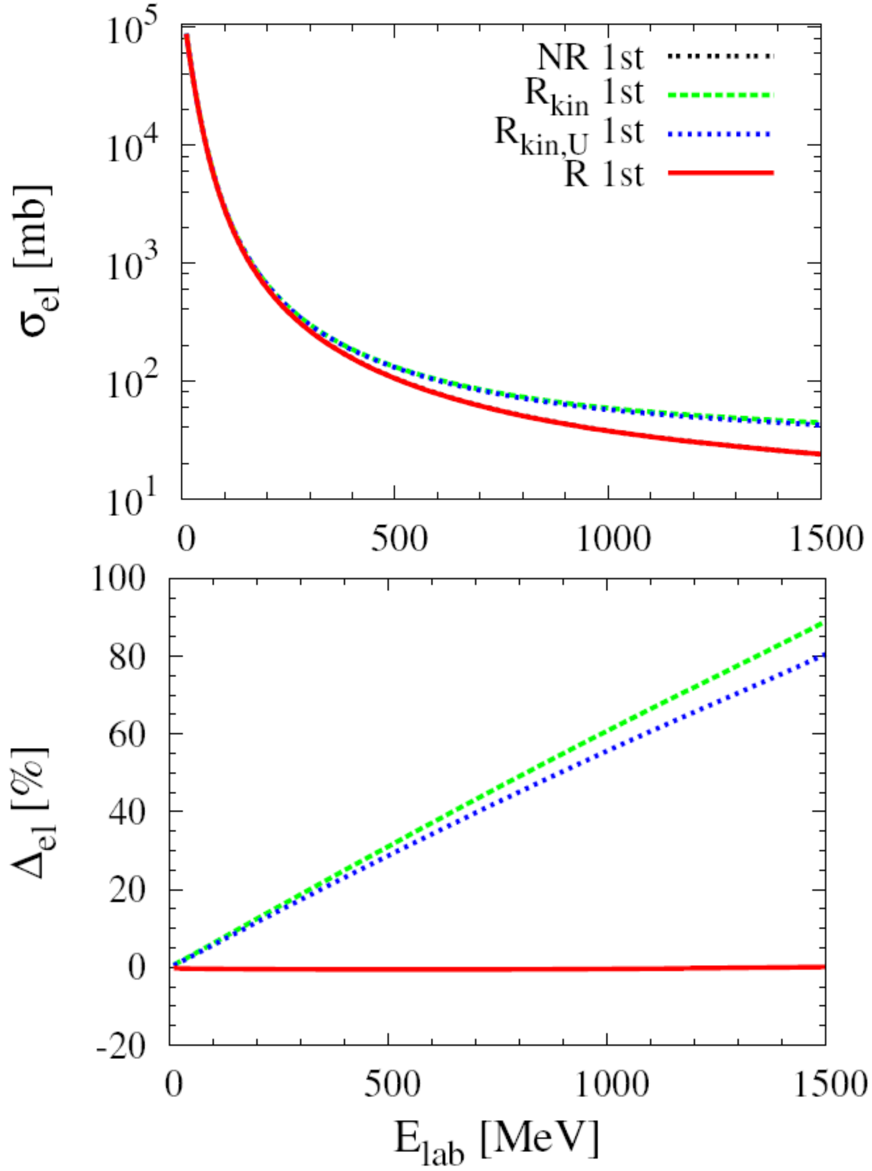
Relativistic kinematics

IA (1^{st} order)

$$T = tP$$

$$U = PG_0^{-1} + PT$$

- Lorentz transformation Lab → c.m. frame) (3-body
- Phase space factors in cross sections
- Poincaré-Jacobi moment
- Permutations



Quantum Mechanics

Galilei Invariant: $H = \frac{\mathrm{K}^2}{2M_g} + h \quad ; \quad h = h_0 + v_{12}^{NR} + v_{13}^{NR} + v_{23}^{NR}$

Poincaré Invariant: $H = \sqrt{\mathrm{K}^2 + M^2} \quad ; \quad M = M_0 + V_{12} + V_{23} + V_{31}$

$$V_{ij} = M_{ij} - M_0 = \sqrt{\left(m_{0,ij} + v_{ij}\right)^2 + q_k^2} - \sqrt{m_{0,ij}^2 + q_k^2}$$

Two-body interaction embedded in the 3-particle Hilbert space

$$m_{0,ij} = \sqrt{m_i^2 + p_{ij}^2} + \sqrt{m_j^2 + p_{ij}^2}$$

$$M_0 = \sqrt{m_{0,ij}^2 + q_k^2} + \sqrt{m_k^2 + q_k^2}$$

V_{ij} embedded in the 3-particle Hilbert space

$$V_{ij} = M_{ij} - M_0 = \sqrt{\left(m_{0,ij} + v_{ij}\right)^2 + q_k^2} - \sqrt{m_{0,ij}^2 + q_k^2}$$

need matrix elements : $\langle \vec{k} | V(\vec{p}) | \vec{k}' \rangle$

$$= v(\vec{k},\vec{k}') + \psi_b(\vec{k})(\sqrt{M_b^2+p^2} - M_b)\psi_b(\vec{k}') + \frac{1}{\omega-\omega'}\Big[(\sqrt{\omega^2+p^2}-\omega)\Re[t(\vec{k}',\vec{k};\omega)]$$

$$-(\sqrt{\omega'^2+p^2}-\omega')\Re[t(\vec{k},\vec{k}';\omega')]\Big] + \frac{1}{\omega-\omega'}\Big[\mathcal{P}\int d^3k'' \frac{(\sqrt{\omega''^2+p^2}-\omega'')}{\omega''-\omega} t(\vec{k},\vec{k}'';\omega'')t^*(\vec{k}',\vec{k}'';\omega'')$$

$$-\mathcal{P}\int d^3k'' \frac{(\sqrt{\omega''^2+p^2}-\omega'')}{\omega''-\omega'} t(\vec{k},\vec{k}'';\omega'')t^*(\vec{k}',\vec{k}'';\omega'')\Big].$$

H. Kamada,[1,*] W. Glöckle,[2,†] J. Golak,[2,3,‡] and Ch. Elster[4,§]

PHYSICAL REVIEW C **66**, 044010 (2002)

Two-Body Input: T1-operator embedded in 3-body system

$$T_1(\mathrm{p}',\mathrm{p};\mathrm{q}) = V(\mathrm{p}',\mathrm{p};\mathrm{q}) + \int d^3k'' \frac{V(\mathrm{p}',\mathrm{k}'';\mathrm{q})\, T_1(\mathrm{k}'',\mathrm{p};\mathrm{q})}{\sqrt{(2E(p'))^2+q^2} - \sqrt{(2E(k''))^2+q^2} + i\varepsilon}$$

Do not solve for V!

- Obtain fully off-shell matrix elements T_1(k,k',q) from half shell transition matrix elements by

  Solving a 1st resolvent type equation:

$$T_1(q) = T_1(q') + T_1(q)\,[g_0(q) - g_0(q')]\,T_1(q')$$

- For every single off-shell momentum point
- Proposed in
  - Keister & Polyzou, PRC 73, 014005 (2006)
- Carried out for the first time here **[PRC 76, 1014010 (2007)]**

Exact Boost

Obtain embedded 2N t-matrix T_1(k,k',z') half-shell in 2-body c.m. frame first :

$$\begin{aligned}\langle \mathbf{k}|T_1(\mathbf{q};z')|\mathbf{k}'\rangle &= \langle \mathbf{k}|V(\mathbf{q})|\mathbf{p}'^{(-)}\rangle \\ &= \frac{2(E_{k'}+E_k)}{\sqrt{4E_{k'}^2+\mathbf{q}^2}+\sqrt{4E_k^2+\mathbf{q}^2}}\, t(\mathbf{k},\mathbf{k}';2E_{k'})\end{aligned}$$

$$t(\mathbf{k},\mathbf{k}';2E_{k'}) = v(\mathbf{k},\mathbf{k}') + \int d\mathbf{k}'' \frac{v(\mathbf{k},\mathbf{k}'')t(\mathbf{k}'',\mathbf{k}';2E_{k'})}{E_{k'}-2\sqrt{m^2+k''^2}+i\epsilon}$$

Solution of the relativistic 2N LS equation with 2-body potential

Consideration for two-body t-matrix

- Relativistic and non-relativistic t-matrix should give identical observables for determining relativistic effects
- Or two-body t-matrices should be phase-shift equivalent
- Four options:
  - Start from relativistic LS equation
    - natural option – employed for NN interactions fit to 1 GeV +
  - **If non-relativistic LS equation is used:**
    - Refit of parameters (maybe time consuming in practice)
    - Transformation of Kamada-Glöckle PRL 80, 2547 (1998)
    - **Transformation of Coester-Piper-Serduke as given in Polyzou PRC 58, 91 (1998)**

Phase equivalent 2-body t-matrices:

Coester-Pieper-Serduke (CPS) (PRC11, 1 (1975))

- Add interaction to square of non-interacting mass operator

$$M^2 = M_0^2 + u = 4mh \quad \text{with } h \equiv \frac{k^2}{m} + \frac{u}{4m} + m$$

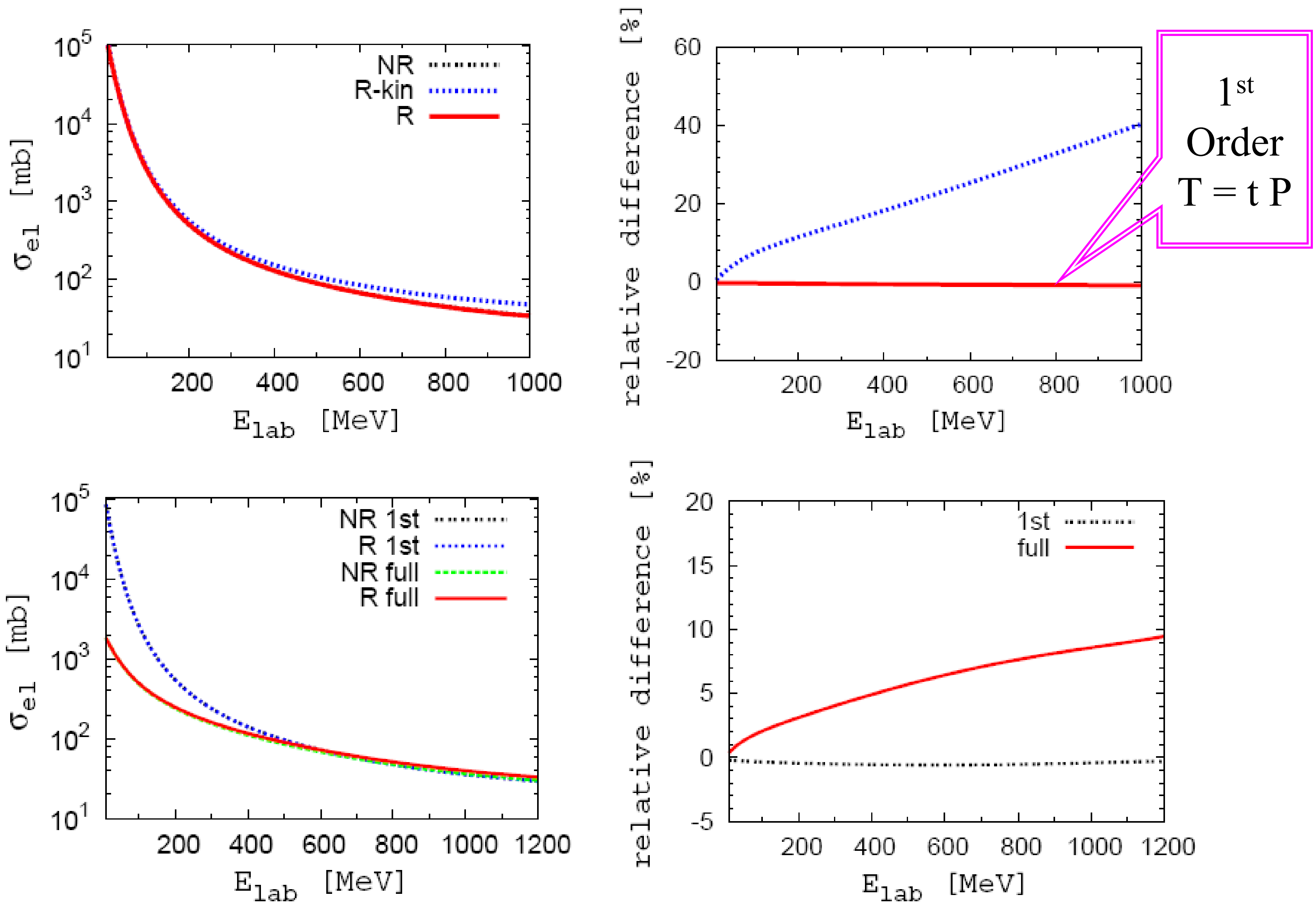
$$u = v^2 + \left\{ M_0^2, v \right\}$$

- NO need to evaluate v directly, since M, M^2, h have the same eigenstates
- Relation between half-shell t-matrices

$$\langle k' | t_R(e(k)) | k \rangle = \frac{4m}{e(k) + e(k')} \langle k' | t_{NR}(k^2/m) | k \rangle$$

- Relativistic and nonrelativistic cross sections are identical functions of the invariant momentum k

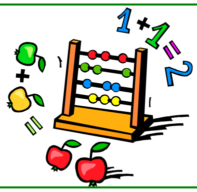
Total Cross Section for Elastic Scattering:



Unitarity Relation

$$\langle \phi | U | \phi' \rangle^* - \langle \phi' | U | \phi \rangle = \int d^3 \mathrm{q} \langle \phi | U | \phi' \rangle^* \, 2\pi \, i \, \vartheta\!\left(E - E_q\right) \langle \phi_q | U | \phi \rangle$$
$$+ \tfrac{1}{3} \int d^3 \mathrm{p} d^3 \mathrm{q} \langle \phi_0 | U_0 | \phi' \rangle \, 2\pi \, i \, \vartheta\!\left(E - E_{pq}\right) \langle \phi_0 | U_0 | \phi \rangle$$

$$-16\pi^3 \frac{E_n(q_0) E_d(q_0)}{q_0 W} \mathrm{Im} \langle q_0, 1, \varphi_d | U | q_0 \varphi_d \rangle = \sigma_{tot} = \sigma_{el} + \sigma_{br}$$



All calculations use a

Malfliet-Tjon type potential

Total Cross Section and Unitarity Relation

E_{lab} GeV	σ_{op} [mb]	σ_{tot} [mb]	σ_{el} [mb]	σ_{br} [mb]
0.1	349.4	350.6	273.4	77.2
0.2	195.1	194.6	158.6	36.0
0.5	106.2	106.8	72.2	34.6
0.8	74.2	74.5	46.6	27.9
1.0	62.3	61.8	37.7	24.1
1.2	54.6	55.3	33.0	22.3
1.5	43.7	44.9	26.0	18.9
2.0	33.0	34.0	18.9	15.2

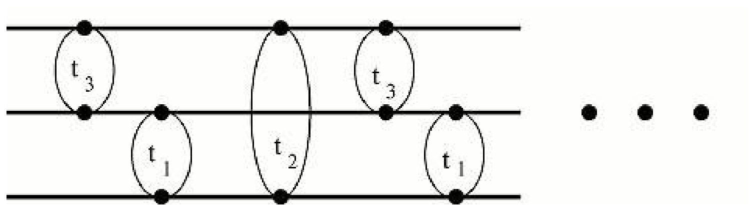
$$\sigma_{tot} = \sigma_{el} + \sigma_{br}$$

Faddeev Equation as multiple scattering series

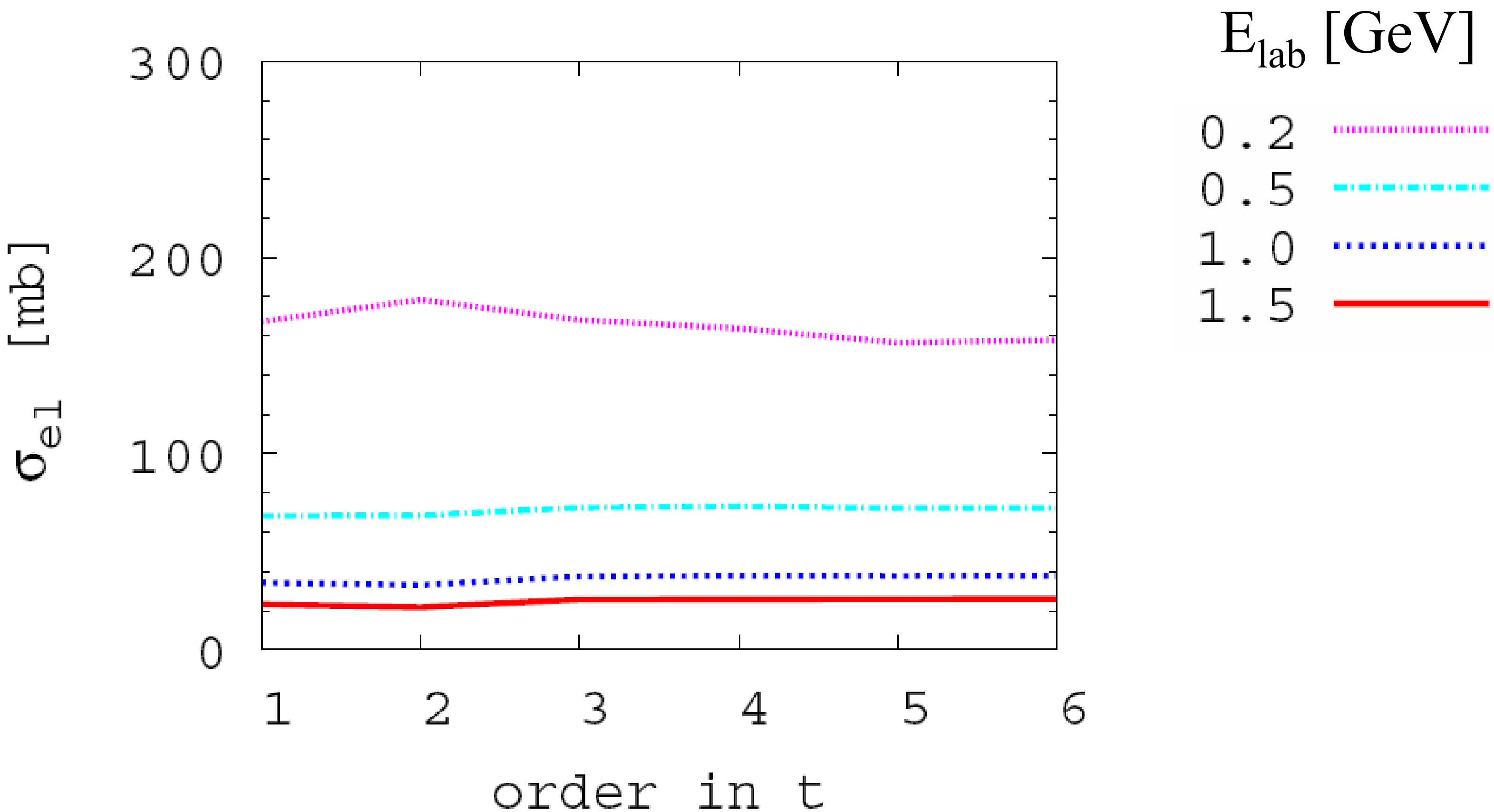
$$T = tP + tG_0PT$$

$$T = tP + tG_0PtP + \cdots$$

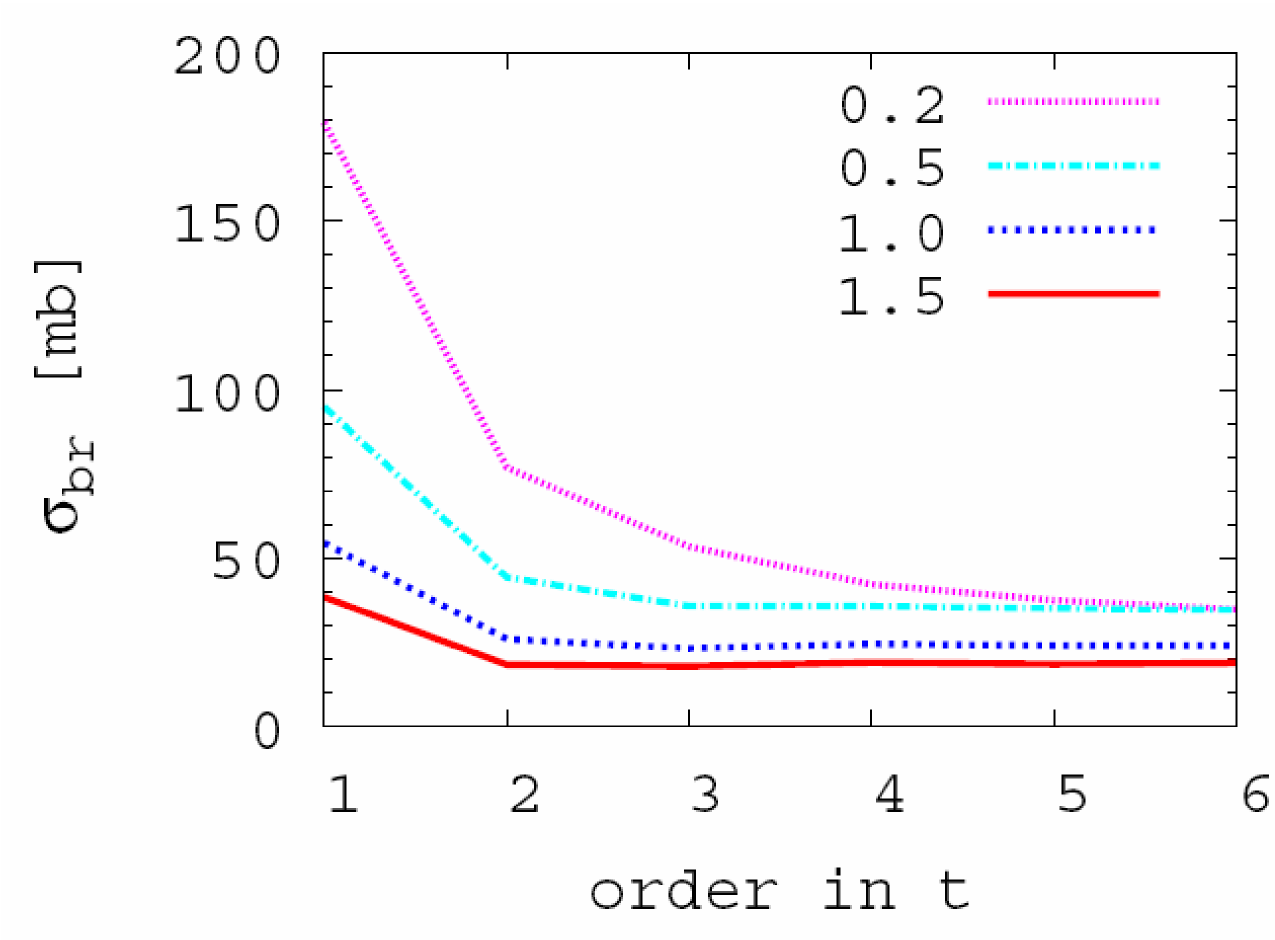
1st Order or IA



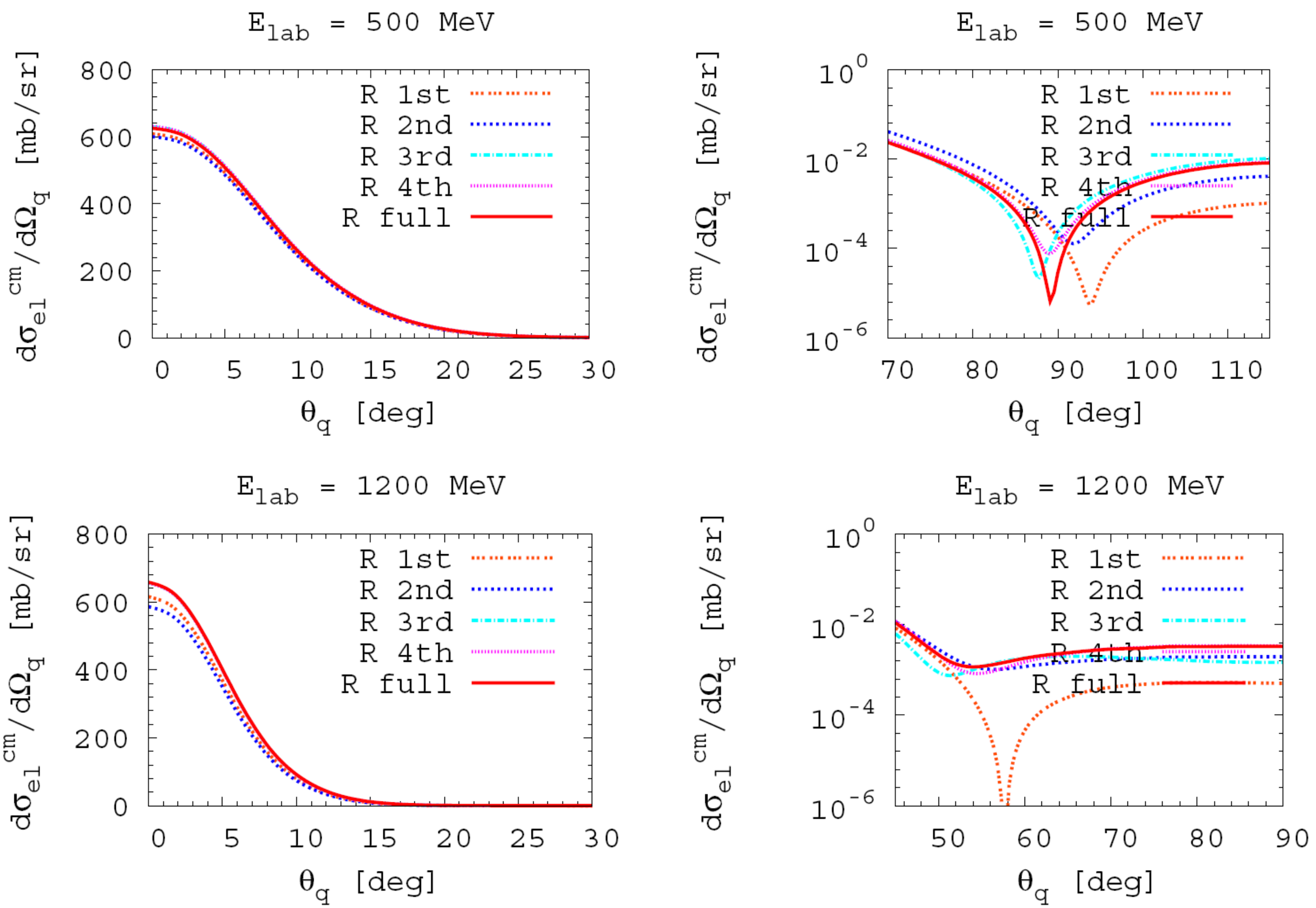
Convergence of the Faddeev Multiple Scattering Series



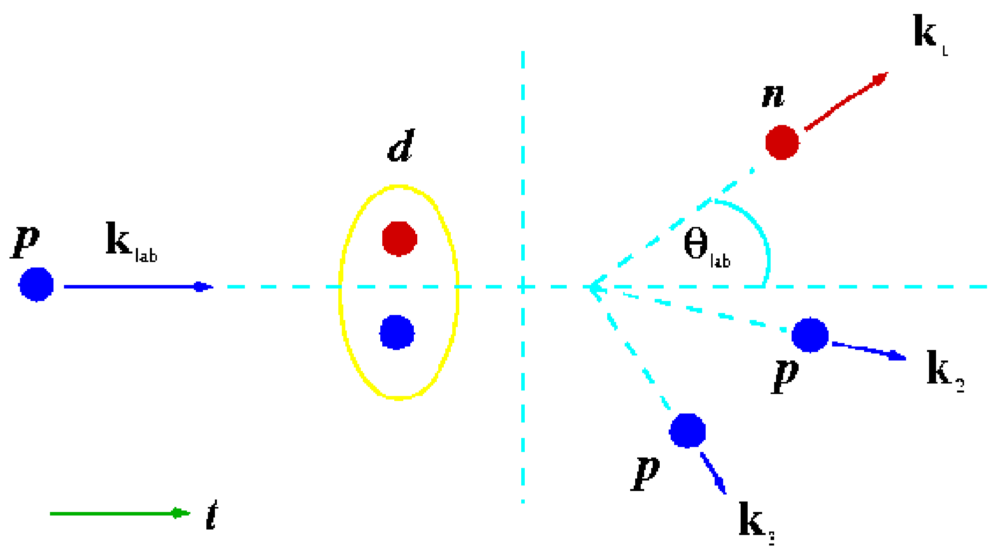
Convergence of the Faddeev Multiple Scattering Series



Elastic Scattering: Differential Cross Section



Breakup Scattering



Exclusive: Measure energy & angles of two ejected particles

V.Punjabi et al. PRC 38, 2728 (1998) – TRIUMF p+d @ 508 MeV

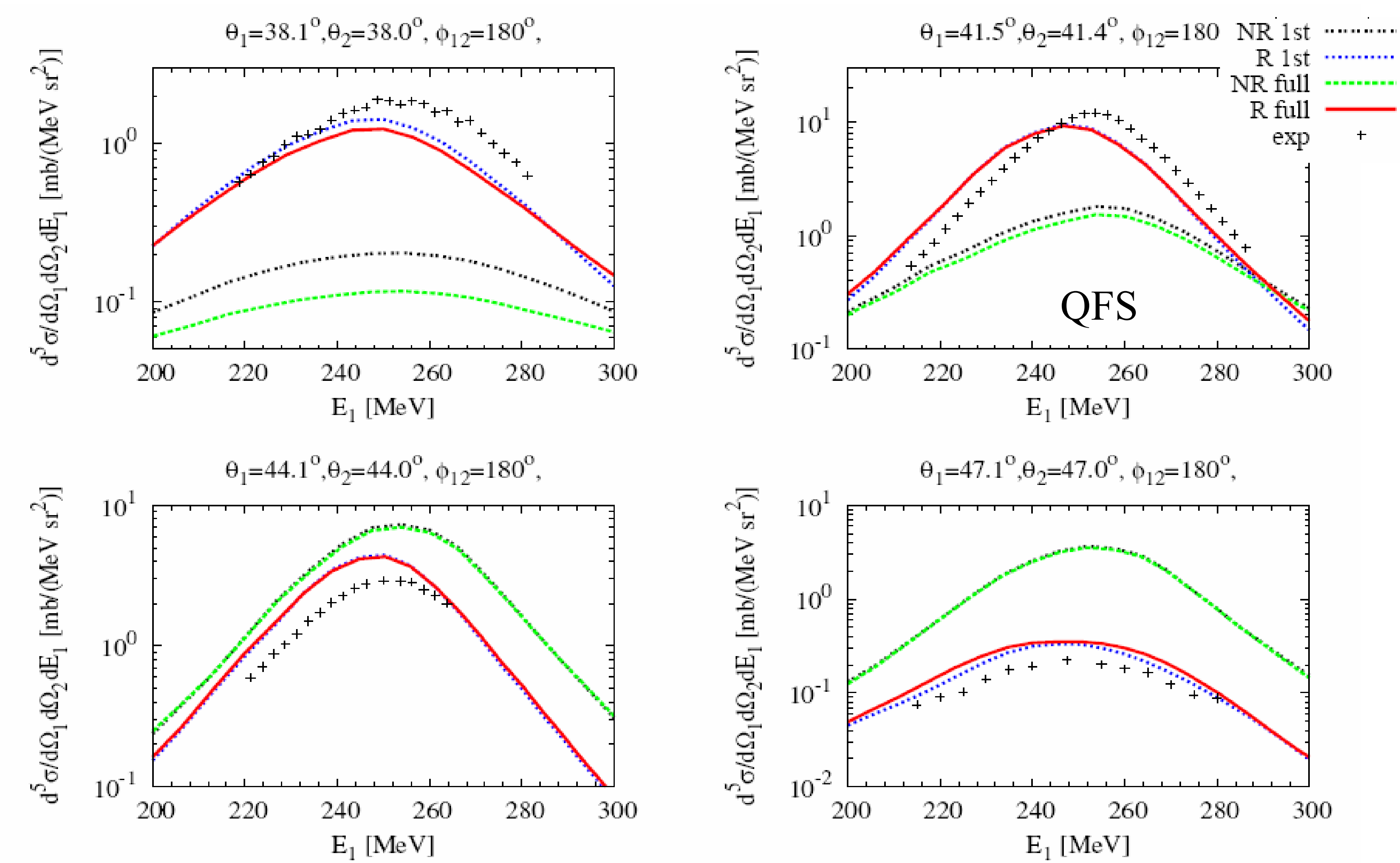
Outgoing protons are measured in the scattering plane

Exclusive Breakup Scattering

(symmetric configuration)

E_{lab} = 508 MeV

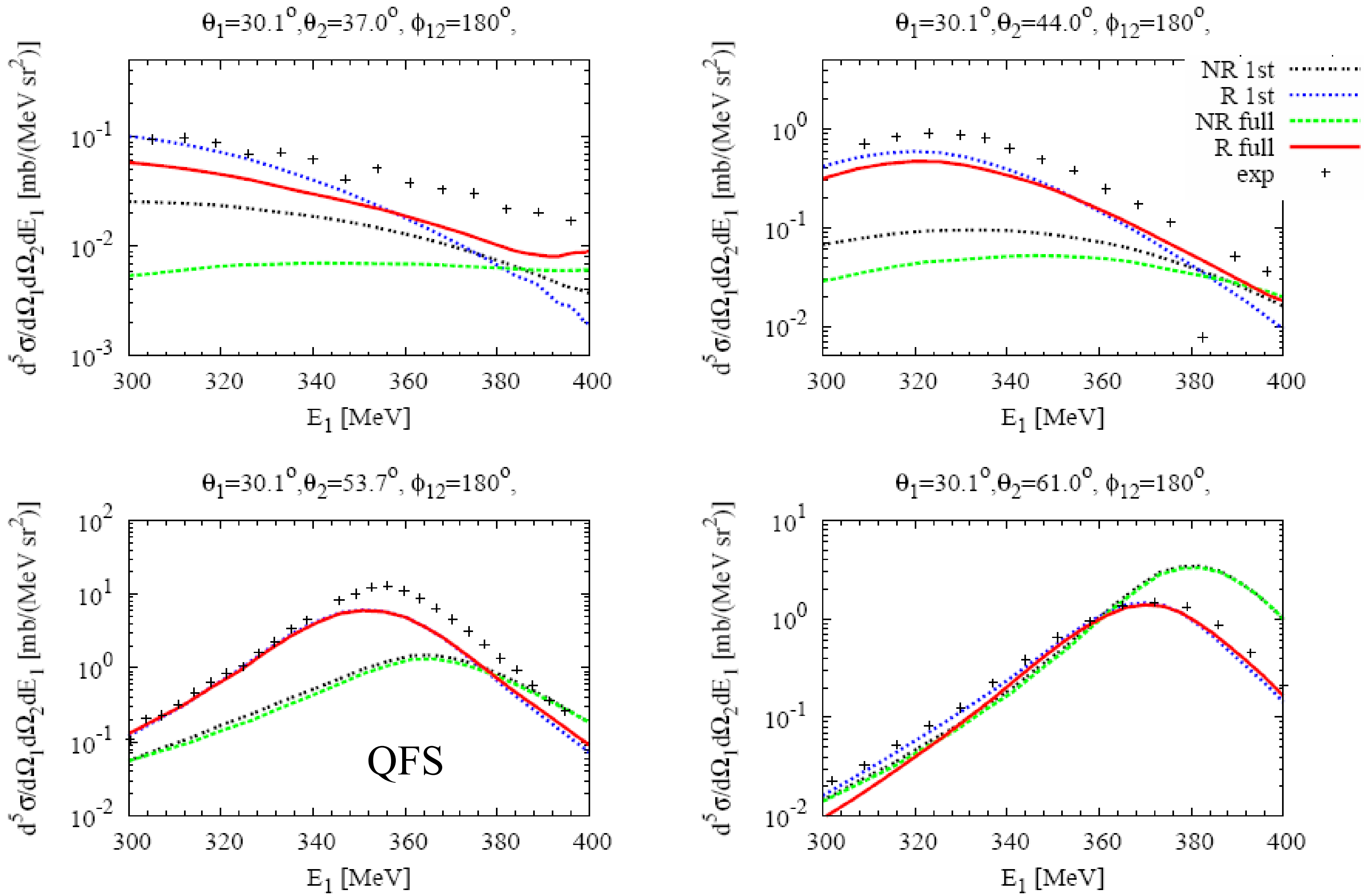
(V.Punjabi et al. PRC 38, 2728 (1998)



Exclusive Breakup Scattering

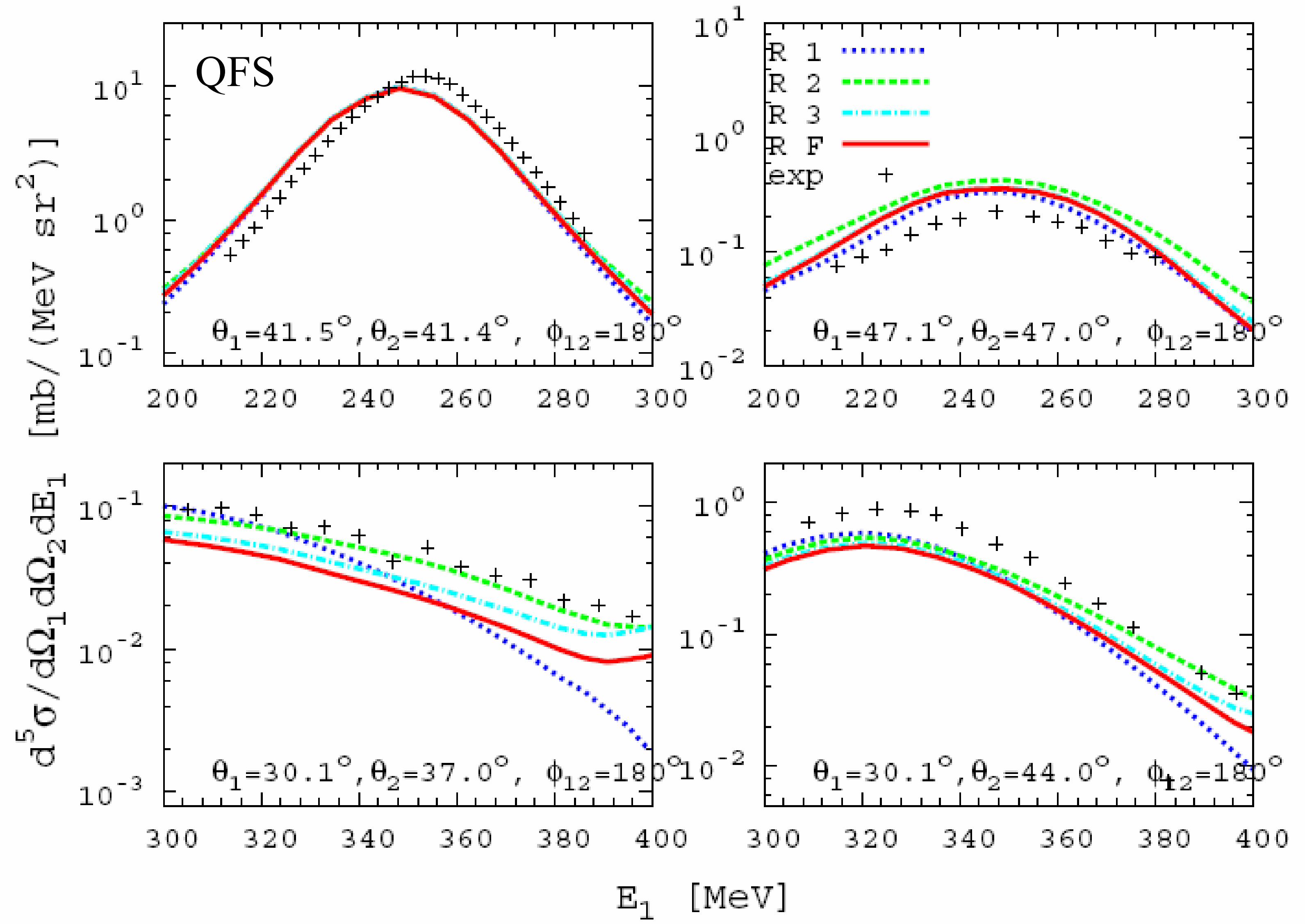
(asymmetric configuration)

E_{lab} = 508 MeV

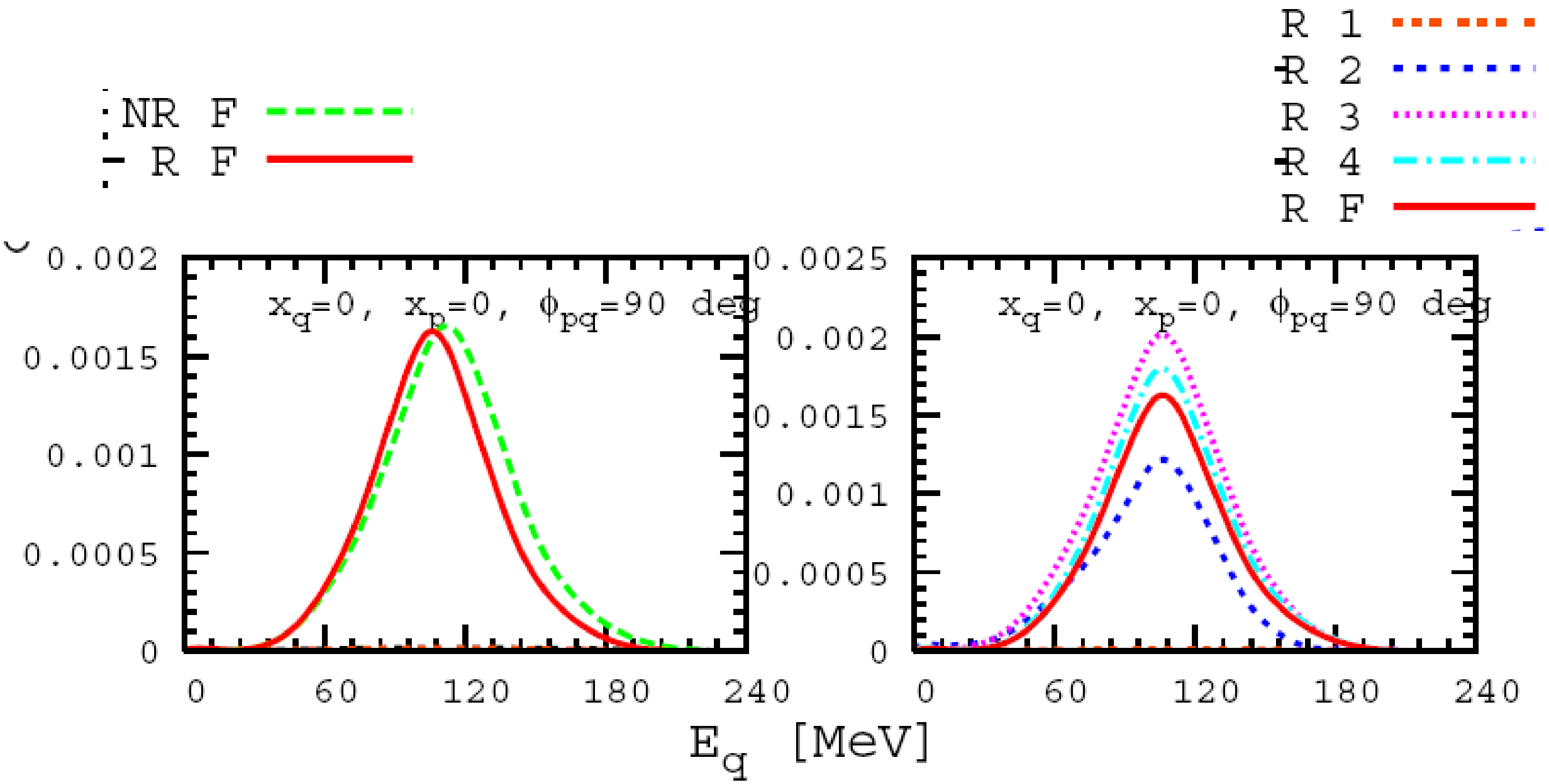


Exclusive Breakup Scattering

E_{lab} = 508 MeV



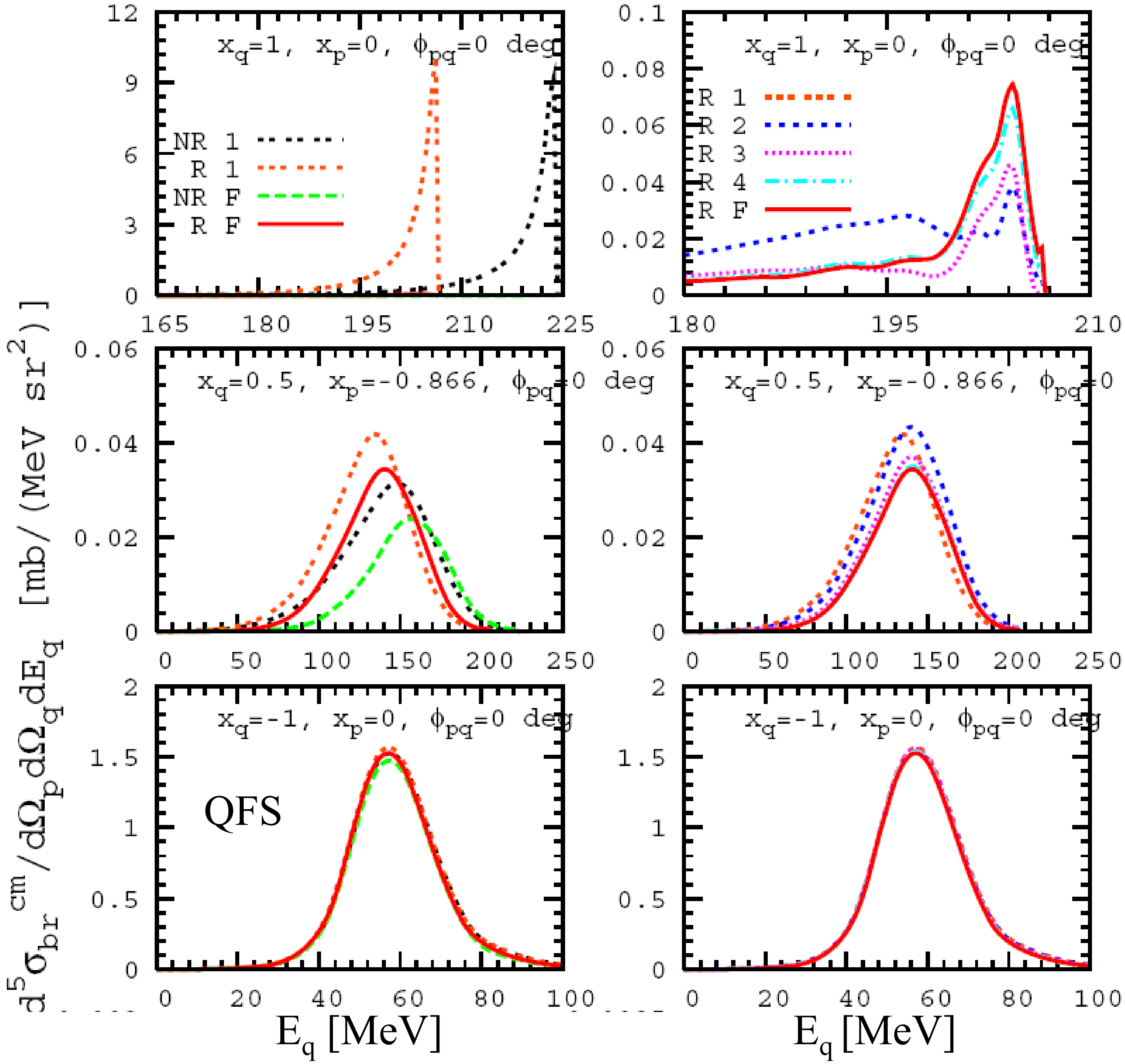
Exclusive Breakup Scattering
Space-Star



$E_{lab} = 508$ MeV

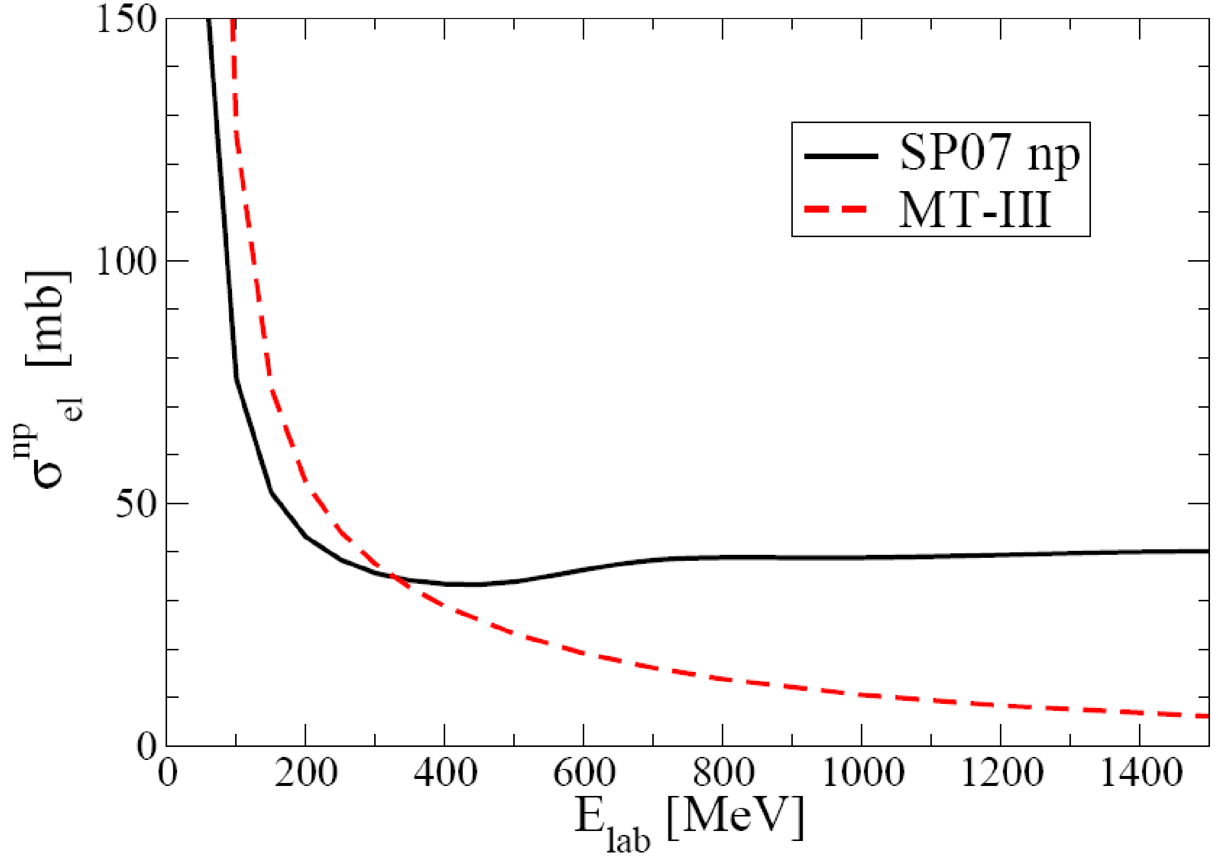
Exclusive Breakup Scattering :

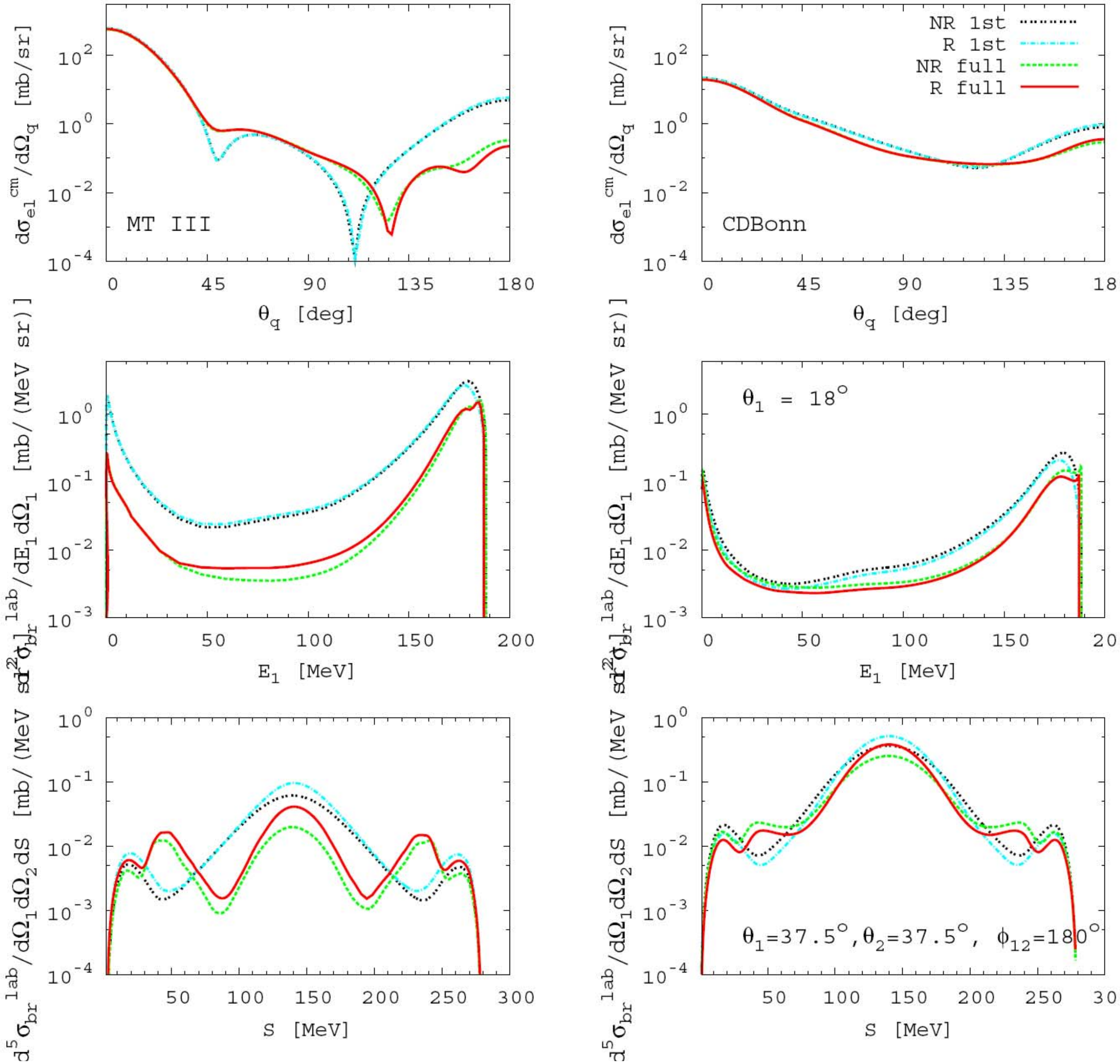
Coplanar Star



E_{lab}= 508 MeV

Relevance of Study with Model Interaction





Calculation:

Henryk Witala

200 MeV

Results for Triton Binding Energy

H.Kamada*[1], W. Glöckle[2], H. Witała[3], J. Golak[3], R. Skibiński[3] and W. N. Polyzou[4] (EFB 20)

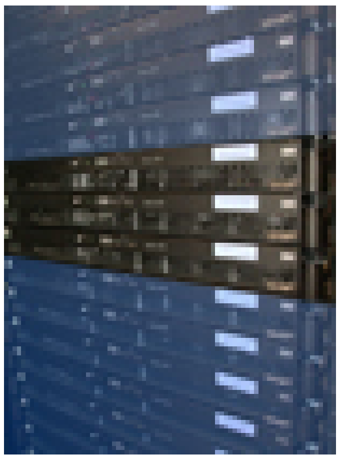
Potential	E_b^{nr}	$E_b^{(1)}$	$\Delta^{(1)}$	$E_b^{(2)}$	$\Delta^{(2)}$
RSC	-7.02	-6.97	0.05	-6.59	0.43
CD-Bonn	-8.33	-8.22	0.11	-7.98	0.35
Nijmegen II	-7.65	-7.58	0.07	-7.22	0.43
Nijmegen I	-8.00	-7.90	0.10	-7.71	0.29
Nijmegen 93	-7.76	-7.68	0.08	-7.46	0.30
AV18	-7.66	-7.59	0.07	-7.23	0.43
Exp. (-8.48)					
		CPS		KG	

5-Channel Calculation

Triton Binding Energy with CD-Bonn (arXiv:0810.2148)

	NR	R	Δ
5-ch (s-wave)	-8.331	-8.219	0.112
18-ch (jm=2)	-8.220	-8.123	0.107
26-ch (jm=3)	-8.241	-8.143	0.098
34-ch (jm=4)	-8.247	-8.147	0.100
34-ch np+nn	-8.005	-7.916	0.089
34-ch (np+nn+wigner)		-7.914	

Computational Equipment



IBM Cluster 1350
970 dP AMD Opteron
(22 TFlop)

Jacquard: 356 dP Opteron Cluster

256 dP Itanium 2 Cluster

Poincaré Invariant Faddeev Calculations

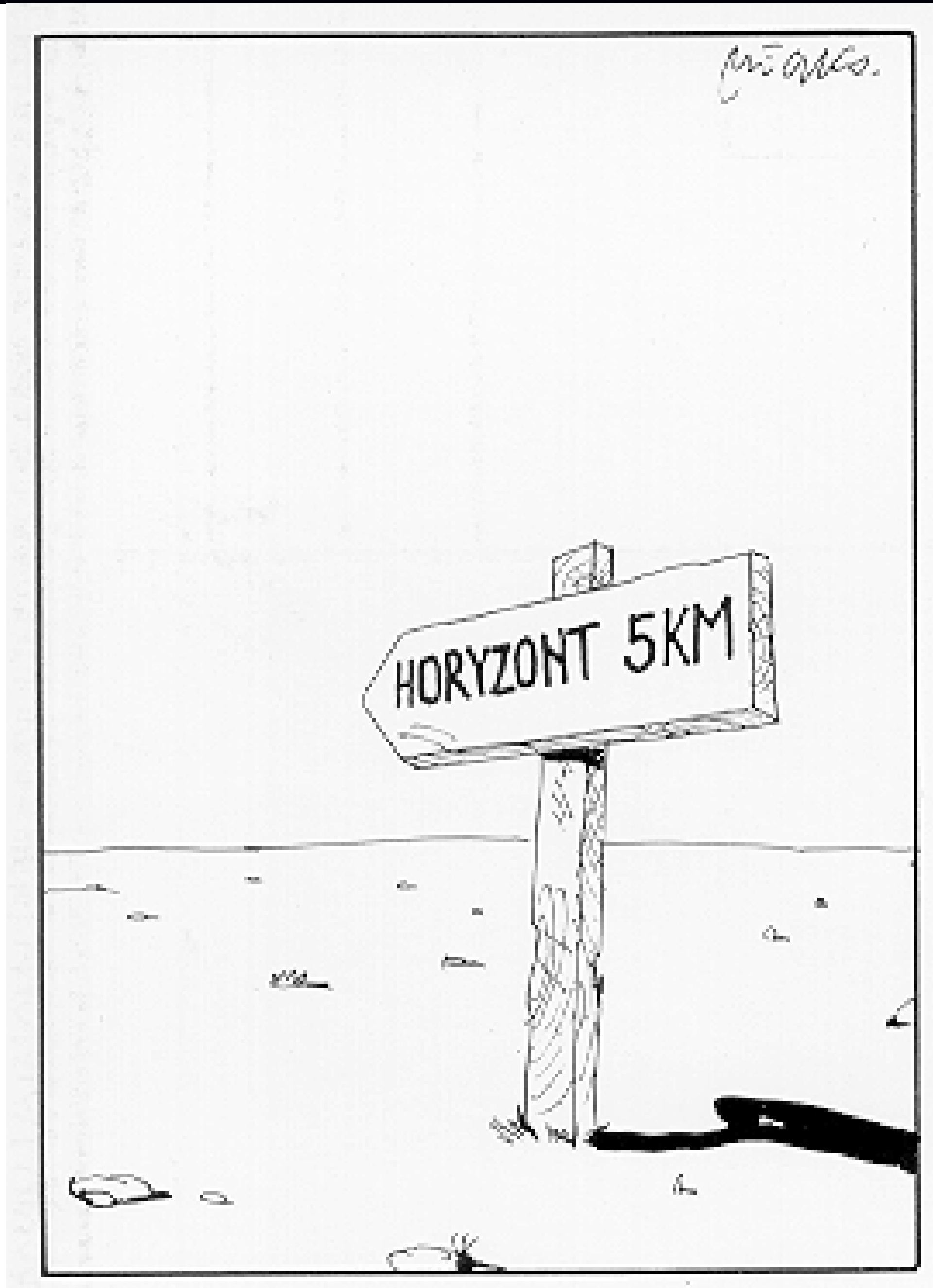
- Kinematics
  - Phase space factors
  - Lorentz Transformation from Lab to c.m. frame
  - Lorentz Transformation of Jacobi Coordinates
    - Always reduces effects of phase-space factors
  - Kinematics determines peak positions in break-up observables
- Dynamics
  - Exact calculation of the two-body interaction embedded in the three-particle Hilbert space
  - The dynamic effects act in general opposite kinematic effects

Poincaré Invariant Faddeev Calculations

- **Carried out up to 2 GeV for elastic and breakup scattering**
  - Solved Faddeev equation in vector variables = NO partial waves
- **Relativistic effects are important at 500 MeV and higher**
  - Relativistic total elastic cross section increases up to 10% compared to the non-relativistic
  - Relativistic kinematics determines QFS peak positions in inclusive and exclusive breakup
  - Breakup: Relativistic effects very large dependent on configuration
- **Above 800 MeV projectile energy:**
  - multiple scattering series converges after ~2 iterations
  - In breakup QFS conditions 1st order calculations sufficient

Poincaré Invariant Faddeev Calculations

- **Triton calculations:**
  - Difference in binding energy between relativistic and nonrelativistic calculation is $\approx$0.1 MeV
  - Provided the CPS realization of a relativistic interaction is used.
  - CPS is in a Hamiltonian context the correct way
- **Future**
  - Systematic studies of selected cross sections & high energy limits
  - Triton: Question about consistent inclusion of 3NF
  - Long term: include Spin


HORYZONT 5KM