Transport Properties of a Delta-Shell Gas with Long Scattering Lengths

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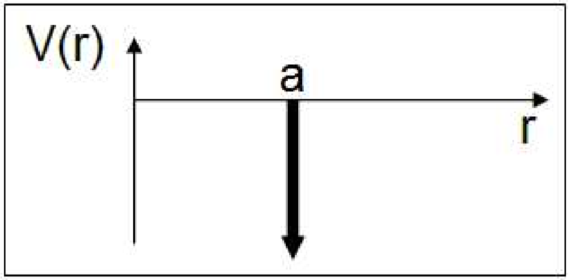
Contents

- Two-particle quantum mechanical inputs
- Scattering lengths & effective ranges
- Bound state properties
- Transport coefficients (viscosity, thermal conductivity & diffusion)
- **Analytical and numerical results**
- **Lessons learned**

The Shrödinger equation

$$\hat{H} = -\hbar^2 \frac{\Delta}{2\mu} - v\,\delta(r-a),$$



$$\hat{H}\,\psi(r)\,Y_{lm}(\theta,\phi) = E\,\psi(r)\,Y_{lm}(\theta,\phi)$$

General solution:

$$\psi(\rho) \equiv \frac{u(\rho)}{\rho} = A_l j_l(\rho) + B_l n_l(\rho)$$

$$\left\{ \begin{array}{l} E = \frac{\hbar^2 k^2}{2\mu} \\ \Lambda = v\frac{2\mu}{\hbar^2} \\ \rho = kr \end{array} \right\}$$

Boundary conditions:

$$u(0) = 0$$

$$\psi(ka-0) = \psi(ka+0)$$

$$\psi'(ka+0) - \psi'(ka-0) = -\frac{\Lambda}{k}\psi(ka)$$

Scattering and phase shifts

Partial wave phase shifts:

$$\tan(\delta_l) = \frac{g\,x\,j_l^2(x)}{1 + g\,x\,j_l(x)\,n_l(x)}$$

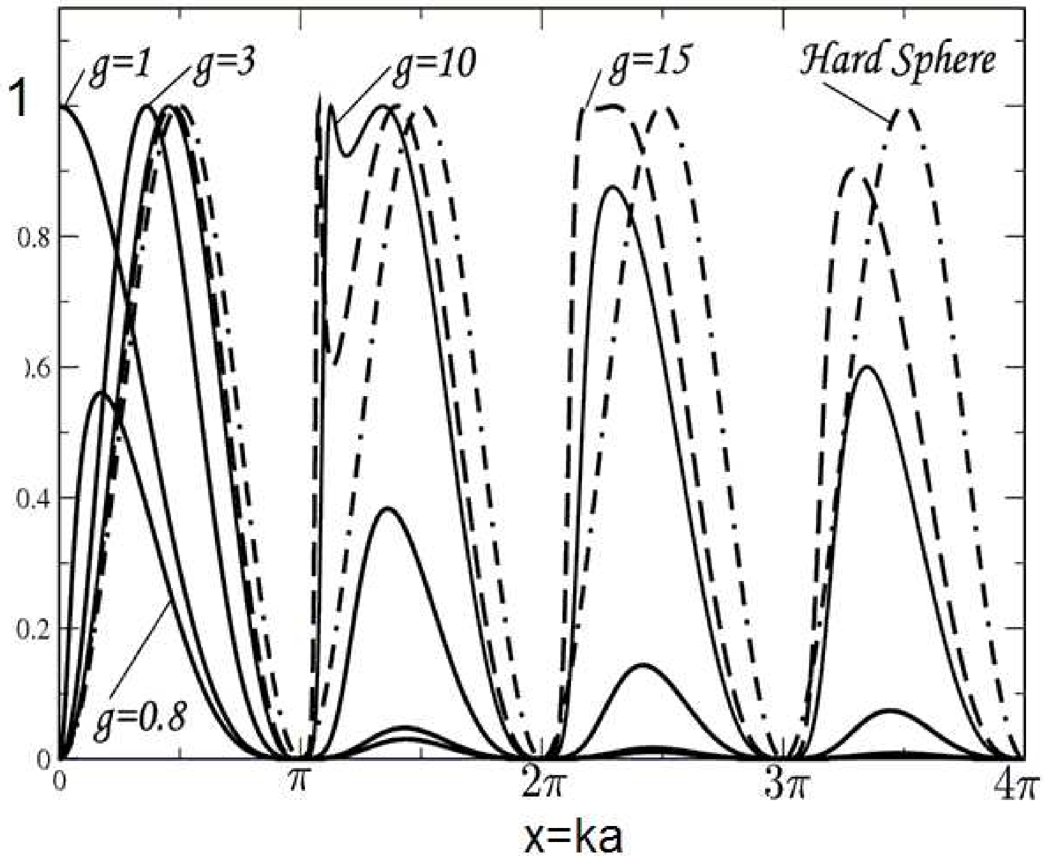
Important dimensionless variables:

$$\begin{cases} x = ka \\ g \equiv \Lambda\, a = \dfrac{2\mu v}{\hbar^2} a \end{cases}$$

Cross section:

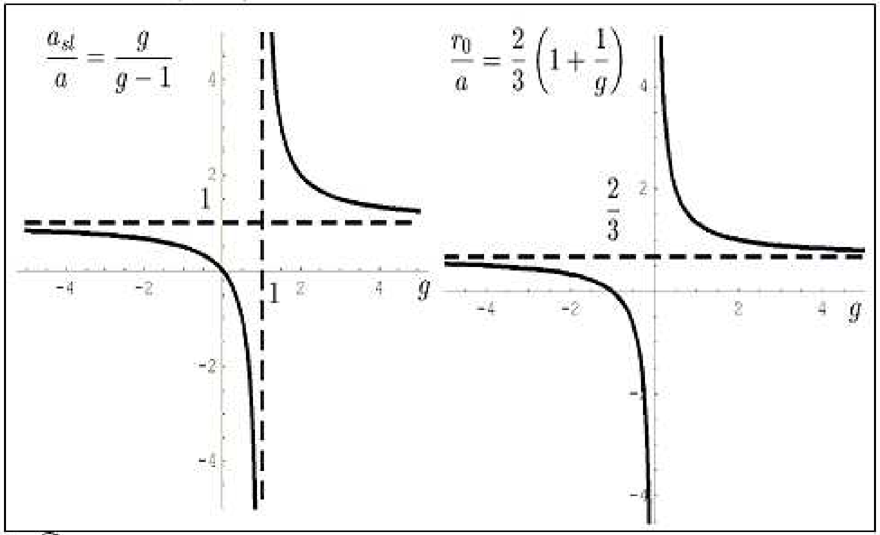
$$\sigma_l(k) \equiv 4\pi a^2 (2l+1) \frac{\sin^2(\delta_l)}{x^2}$$

$$\sigma(k) = \sum_{l=0}^{\infty} \sigma_l(k)$$



Scattering length and range

$$k\cot(\delta_0) = -\frac{1}{a_{sl}} + r_0\frac{k^2}{2} - P\, r_0^3 k^4 + O(k^6)$$



$$\text{scattering length: } \frac{a_{sl}}{a} = \frac{g}{g-1},$$

$$\text{effective range: } \frac{r_0}{a} = \frac{2}{3}\left(1+\frac{1}{g}\right),$$

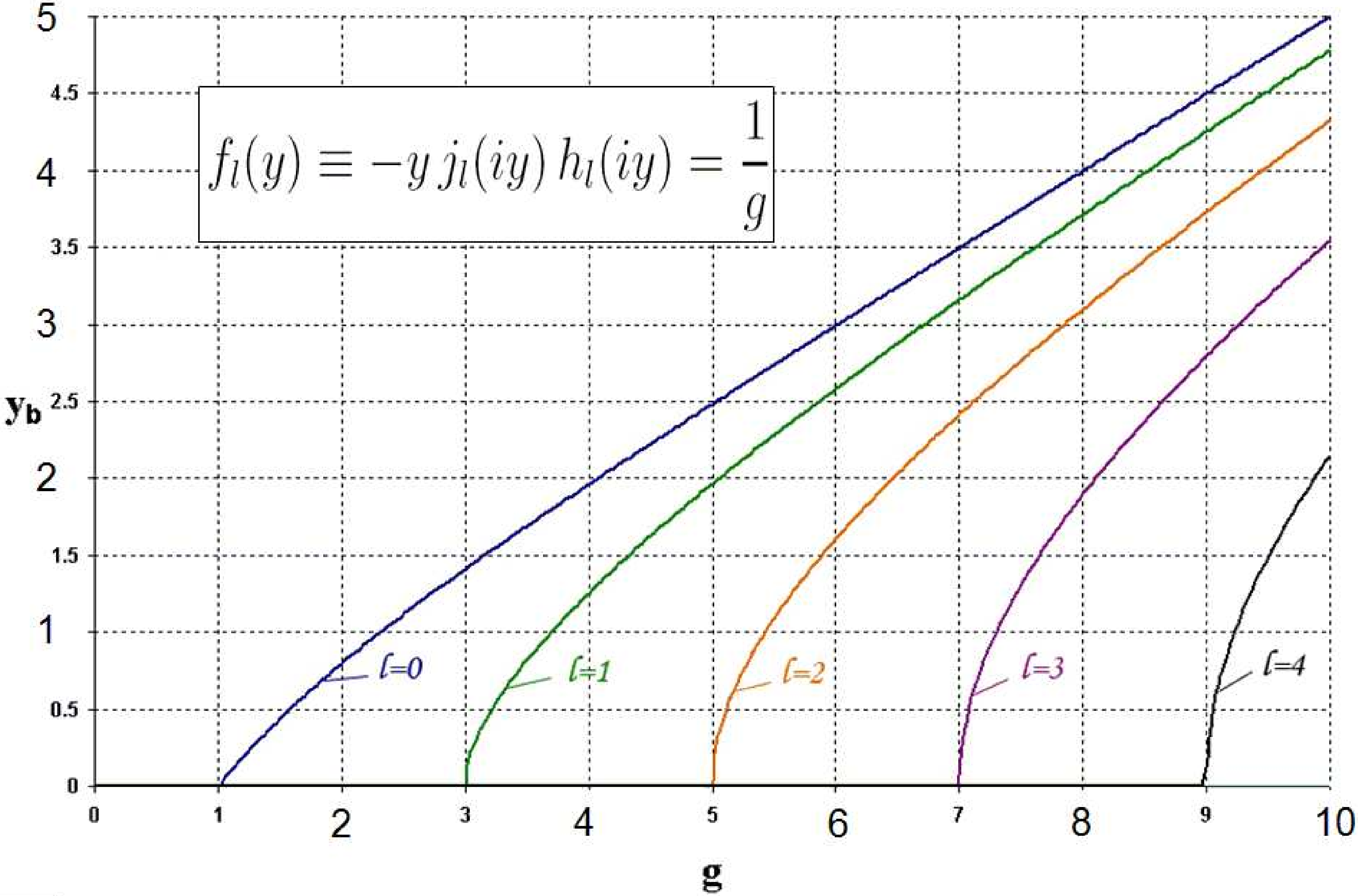
$$\text{shape parameter: } P = -\frac{3}{40}\frac{g^2(3+g)}{(1+g)^3}.$$

$$k^{2l+1}\cot(\delta_l) = -\frac{1}{a_{sl}^{(l)}} + r_0^{(l)}\frac{k^2}{2} + O(k^4)$$

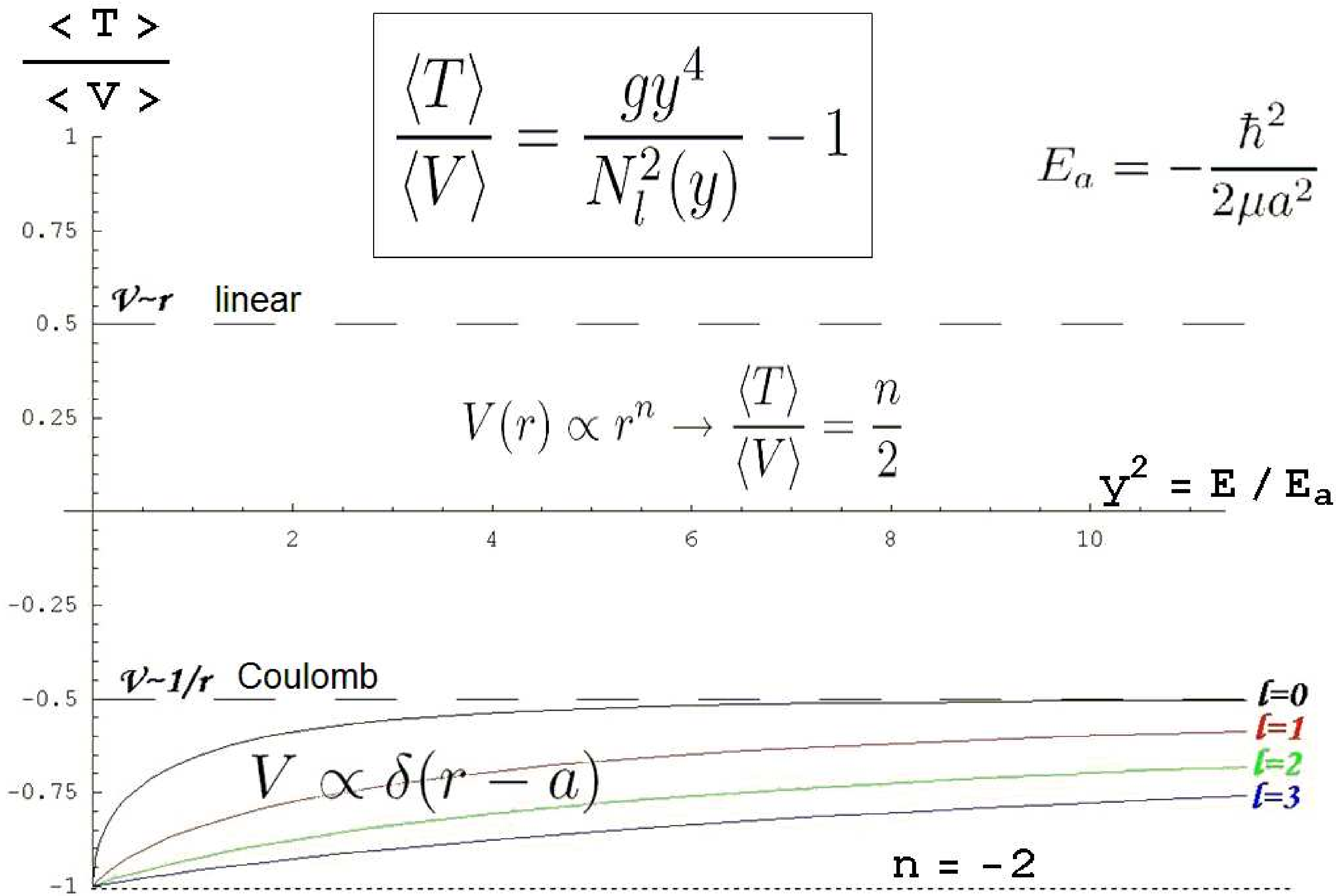
$$\frac{a_{sl}^{(l)}}{a} = \frac{(2l+1)}{((2l+1)!!)^2}\left(\frac{g}{g-(2l+1)}\right),$$

$$\frac{r_0^{(l)}}{a} = \frac{((2l+1)!!)^2}{(2l+3)(2l-1)}\left(\frac{2l-1}{g} - 1\right).$$

Bound states $E = -\frac{\hbar^2 {y_b}^2}{2\mu a^2}$



Virial Theorem for bound states with energy E

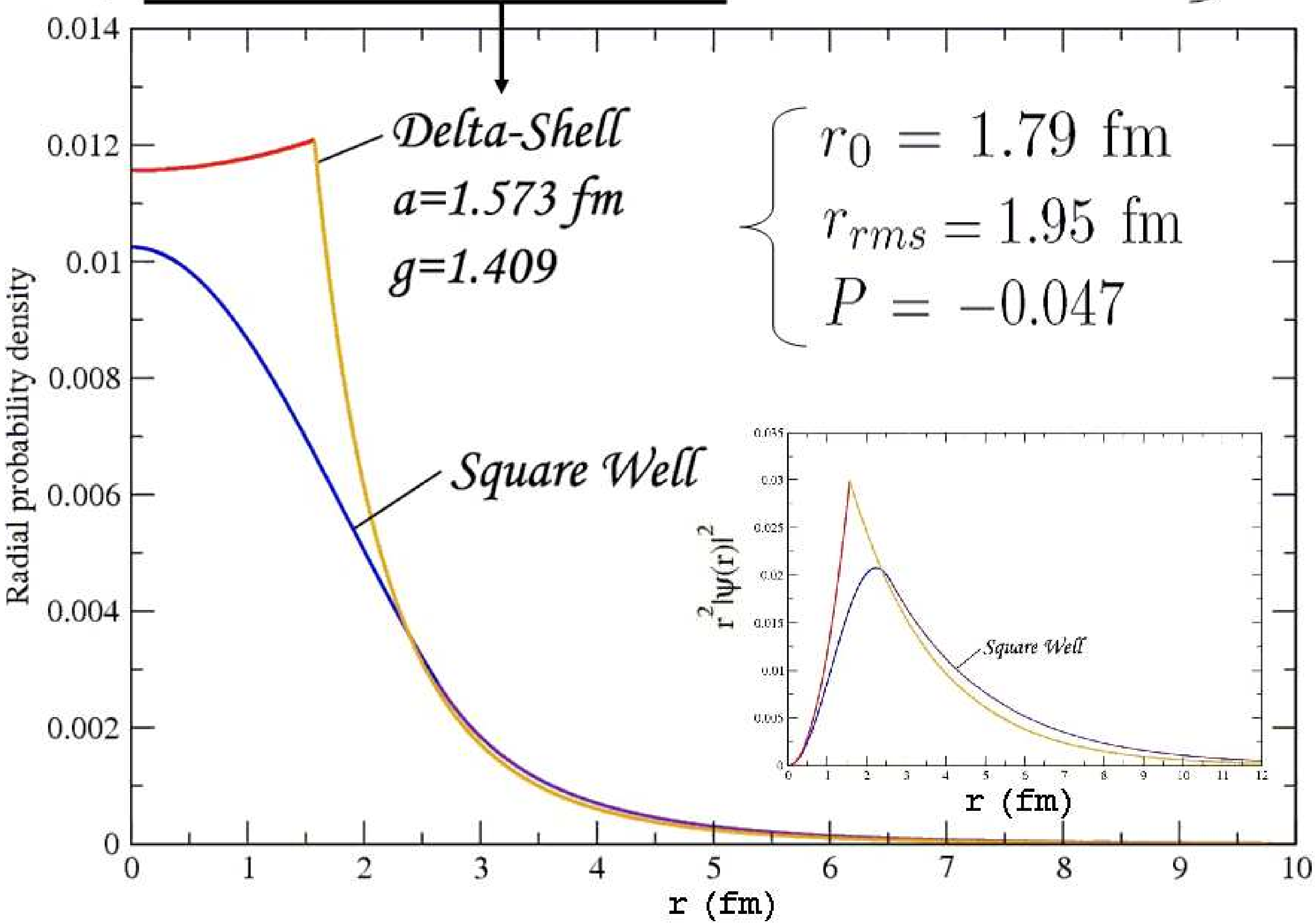


Deuteron:

$$r_0 = 1.76 \text{ fm}$$
$$r_{rms} = 1.97 \text{ fm}$$
$$P = -.007$$

Exp.

a_s=5.52 fm & E=-2.2246 MeV



Some useful scales

Thermal de-Broglie wave length :

$$\lambda(T) = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{1/2}$$

Delta-shell range : a Dilution parameter : na^3

Delta-shell scattering length :

$$a_{sl} = \frac{ag}{g-1} \; ; \qquad \text{Unitary limit} \; : \; g = 1$$

Hard-sphere like transport (diffusion, viscosity & thermal conductivity) coefficients :

$$\tilde{\mathcal{D}} = \frac{3\sqrt{2}}{32} \frac{\hbar}{mna^3} \, , \qquad \tilde{\eta} = \frac{5\sqrt{2}}{32} \frac{\hbar}{a^3} \, , \qquad \text{and} \qquad \tilde{\kappa} = \frac{75}{64\sqrt{2}} \frac{\hbar k_B}{ma^3} \, .$$

Overview of transport properties

Table 14.2 Summary of phenomenological transport laws

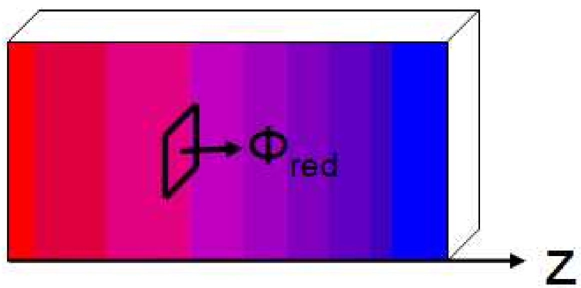
Effect	Flux of particle property	Gradient	Coefficient	Law	Name of law	Approximate expression for coefficient
Diffusion	Number	$\frac{dn}{dz}$	Diffusivity D	$\mathbf{J}_n = -D \operatorname{grad} n$	Fick's law	$D = \frac{1}{3}\bar{c}l$
Viscosity	Transverse momentum	$M\frac{dv_x}{dz}$	Viscosity η	$\frac{F_x}{A} = J_{\mathbf{p}}{}^x = -\eta\frac{dv_x}{dz}$	Newtonian viscosity	$\eta = \frac{1}{3}\rho\bar{c}l$
Thermal conductivity	Energy	$\frac{d\rho_u}{dz} = \hat{C}_V\frac{dT}{dz}$	Thermal conductivity K	$\mathbf{J}_u = -K \operatorname{grad} \tau$	Fourier's law	$K = \frac{1}{3}\hat{C}_V\bar{c}l$
Electrical conductivity	Charge	$-\frac{d\varphi}{dz} = E_z$	Conductivity σ	$\mathbf{J}_q = \sigma\mathbf{E}$	Ohm's law	$\sigma = \frac{nq^2 l}{M\bar{c}}$

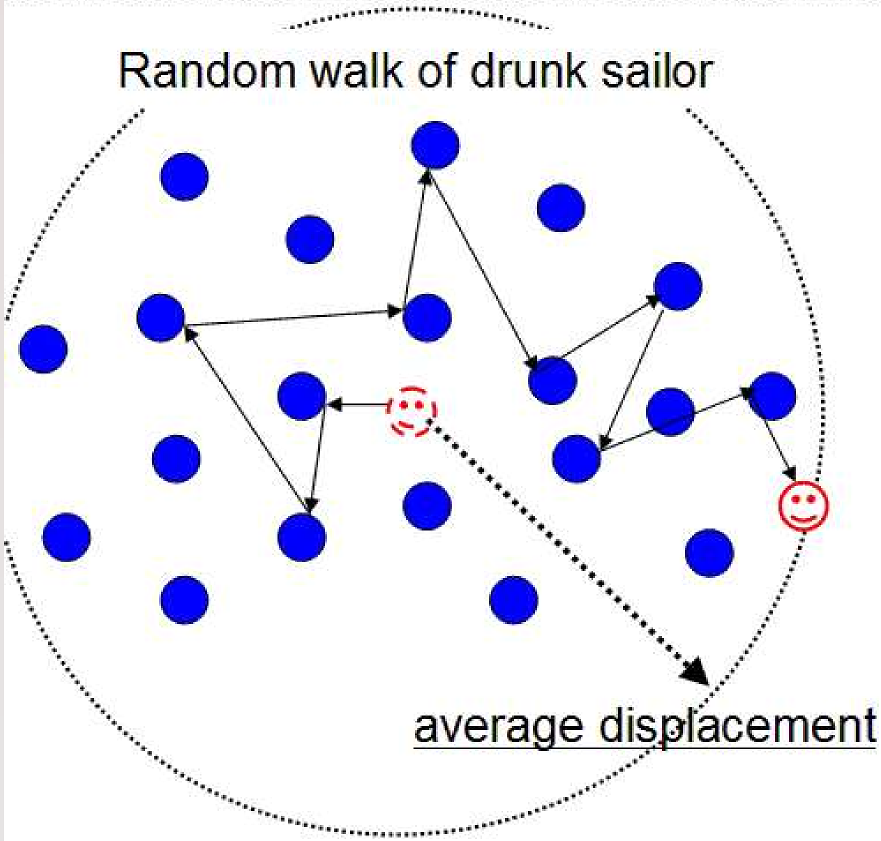
SYMBOLS: n = number of particles per unit volume
$\bar{c}$ = mean thermal speed = $\langle|v|\rangle$
l = mean free path
$\hat{C}_V$ = heat capacity per unit volume
ρ_u = thermal energy per unit volume
F_x/A = shear force per unit area
φ = electrostatic potential
$\mathbf{E}$ = electric field intensity
q = electric charge
M = mass of particle
ρ = mass per unit volume
$\mathbf{p}$ = momentum

"Thermal Physics" Ch. Kittel / H. Kroemer

Diffusion

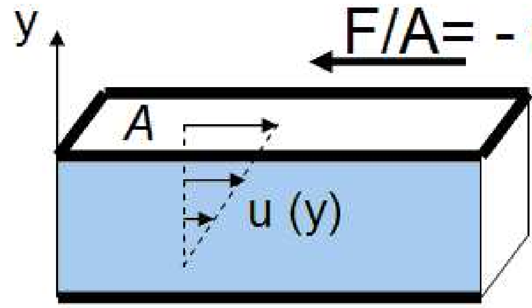
$$\Phi_{red} = -D\, dn_{red}/dz$$



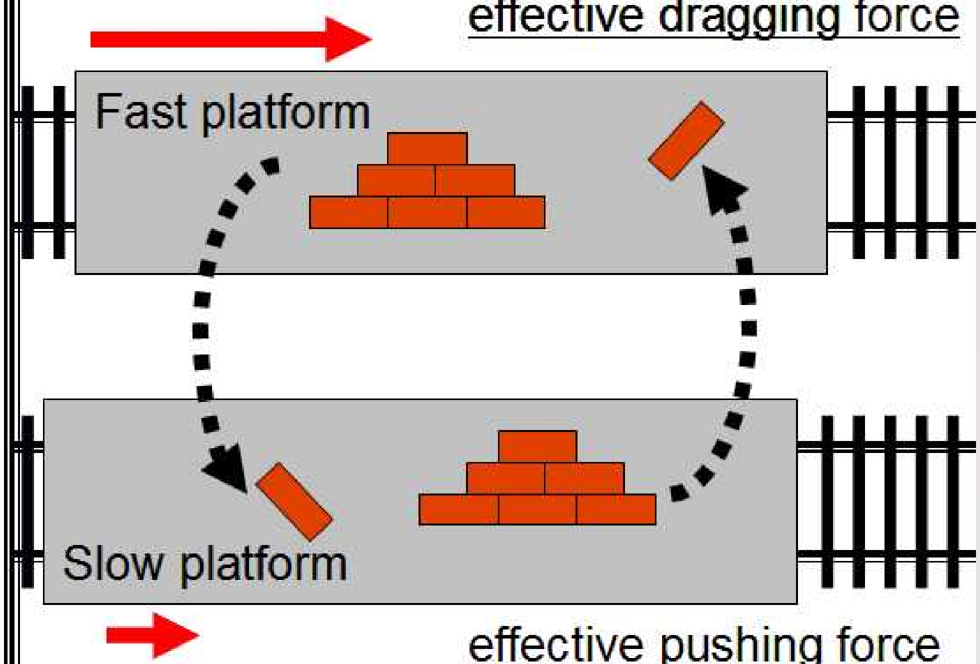


Shear viscosity

$$F/A = -\eta\, du/dy$$

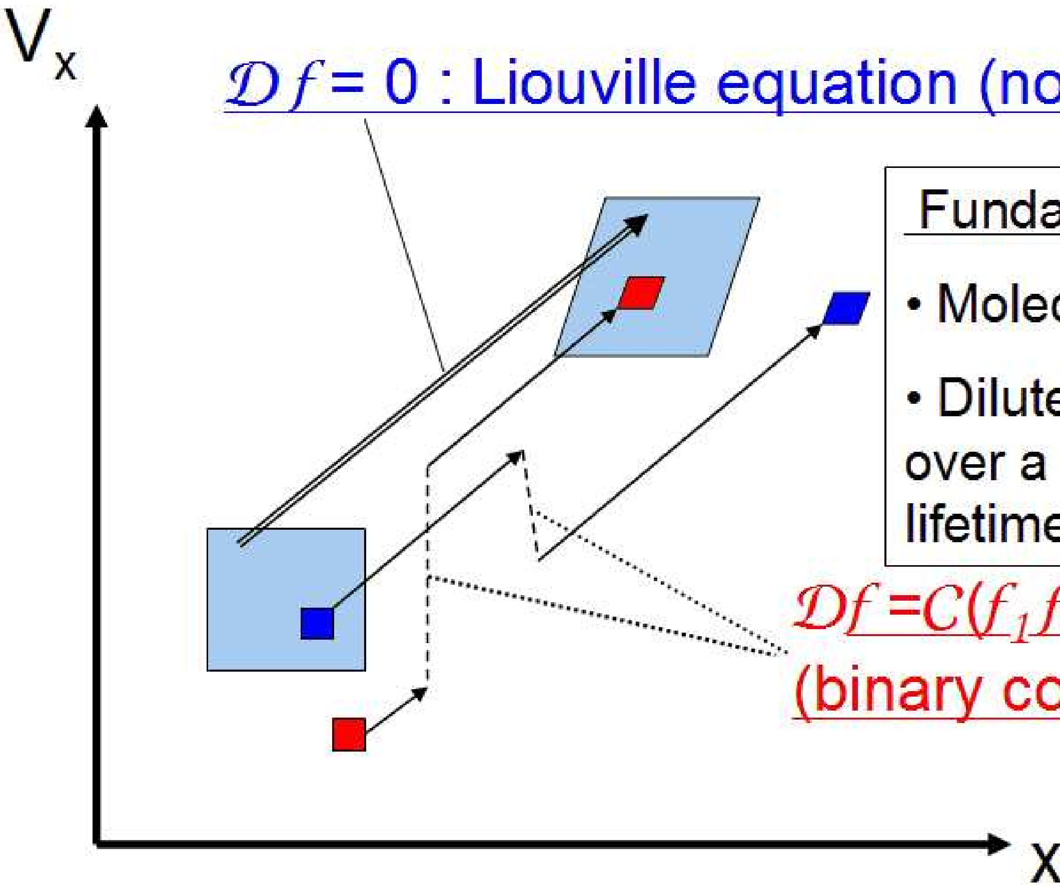


Classical Newtonian laminar flow



Boltzmann equation for the particle distribution function: f (phase space)

Equations of motion → time evolution → operator $\mathcal{D}$

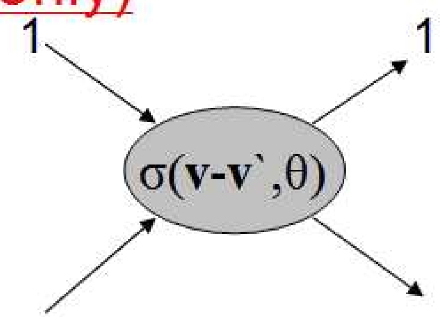


$\mathcal{D} f = 0$: Liouville equation (no collisions)

Fundamental assumptions:

- Molecular chaos – no correlations
- Dilute system – encounters occur over a small fraction of molecular lifetime

$\mathcal{D}f = \mathcal{C}(f_1 f)$: Boltzmann equation (binary collisions only)



The Boltzmann Equation

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p_1}}{m}\cdot\nabla_r + \mathbf{F}\cdot\nabla_{p_1}\right) f_1 \;=\; \int d^3p_2\, d^3p_1'\, d^3p_2'\;\delta^4(P_f - P_i)$$
$$\times\;\; |T_{fi}|^2\,(f_2' f_1' - f_2 f_1)$$

- Nonlinear integro-differential equation for f_1
- Except in rare cases, analytical solutions not available

The collision integral on the right hand side can be cast as

$$C = \int d^3p_2\, d\Omega\, |\mathbf{v_1} - \mathbf{v_2}|\,(d\sigma/d\Omega)\,(f_2' f_1' - f_2 f_1)$$

- Modifications due to Pauli supression or Bose enhancement can also be incorporated

Variables of hydrodynamics

Basic variables: $f \equiv f(\mathbf{r}, \mathbf{v}, t)$

$$\langle A \rangle = \int d^3p \, Af / \int d^3p \, f \qquad \text{(Expectation value of } A\text{)}$$

$$\mathbf{v}(\mathbf{r}, t) = \langle v \rangle \qquad \text{(Average velocity)}$$

$$\rho = m \int d^3v \, f \qquad \text{(mass density)}$$

$$\theta(\mathbf{r}, t) = \frac{1}{3} m \left\langle |\mathbf{v} - \mathbf{u}|^2 \right\rangle \qquad \text{(heat flux)}$$

$$P_{ij} = \rho \left\langle (v_i - u_i)\,(v_j - u_j) \right\rangle \qquad \text{(Pressure tensor)}$$

Dissipative hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \text{(continuity)}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \mathbf{u} = \frac{\mathbf{F}}{m} - \frac{1}{\rho}\left(P - \frac{\eta}{3} \nabla \cdot \mathbf{u}\right) + \frac{\eta}{\rho} \nabla^2 \mathbf{u}$$

$$\text{(Navier} - \text{Stokes equation)}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \theta = -\frac{1}{c_v}(\nabla \cdot \mathbf{u})\theta + \frac{\kappa}{\rho c_v} \nabla^2 \theta$$

$$\text{(Heat conduction)}$$

Enskog's approximate solution of the Boltzmann equation

System is assumed to be only slightly disturbed from the equilibrium state $f^{(0)}$:

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \ldots$$

Boltzmann equation: $F[f] = 0$

$$F[f] = F^{(0)}[f^{(0)}] + F^{(1)}[f^{(0)},f^{(1)}] + F^{(2)}[f^{(0)},f^{(1)},f^{(2)}] + \ldots$$

$F^{(0)}[f^{(0)}] = 0 \rightarrow$ Maxwell distribution

$F^{(1)}[f^{(0)},f^{(1)}] = 0 \rightarrow$ first approximation

$F^{(2)}[f^{(0)},f^{(1)},f^{(2)}] = 0 \rightarrow$ second approximation

…

Transport integrals

Transport cross section $\phi^{(n)} = 2\pi \int_{-1}^{+1} d\cos\theta (1 - \cos^n\theta) \frac{d\sigma(k,\theta)}{d\Omega}\Big|_{c.m.}$

$$q^{(1)} \equiv \frac{\phi^{(1)}}{4\pi a^2} = \frac{2}{x^2} {\sum_l}' (2l+1) \sin^2(\delta_l),$$

$$q^{(2)} \equiv \frac{\phi^{(2)}}{4\pi a^2} = \frac{2}{x^2} {\sum_l}' \frac{(l+1)(l+2)}{(2l+3)} \sin^2(\delta_{l+2} - \delta_l),$$

$$\omega^{(n,t)}(T) \equiv \int_0^\infty d\gamma\, e^{-\gamma^2} \gamma^{2t+3} q^{(n)}(x)$$

$$\gamma = \frac{\hbar k}{\sqrt{2\mu k_B T}} = \frac{x}{\sqrt{2\pi}} \left(\frac{\lambda(T)}{a} \right)$$

Analysis for particles with spin

$$q_{(s)}^{(n)} = \frac{s+1}{2s+1}\, q_{Bose}^{(n)} + \frac{s}{2s+1}\, q_{Fermi}^{(n)}, \quad \text{for integer } s\,,$$

$$q_{(s)}^{(n)} = \frac{s+1}{2s+1}\, q_{Fermi}^{(n)} + \frac{s}{2s+1}\, q_{Bose}^{(n)}, \quad \text{for half-integer } s\,.$$

Here, we will present results for the case of spin-1/2 particles only.

Shear viscosity

$$\tilde{\eta} = \frac{5h}{32\sqrt{2}\pi}\frac{1}{a^3},$$

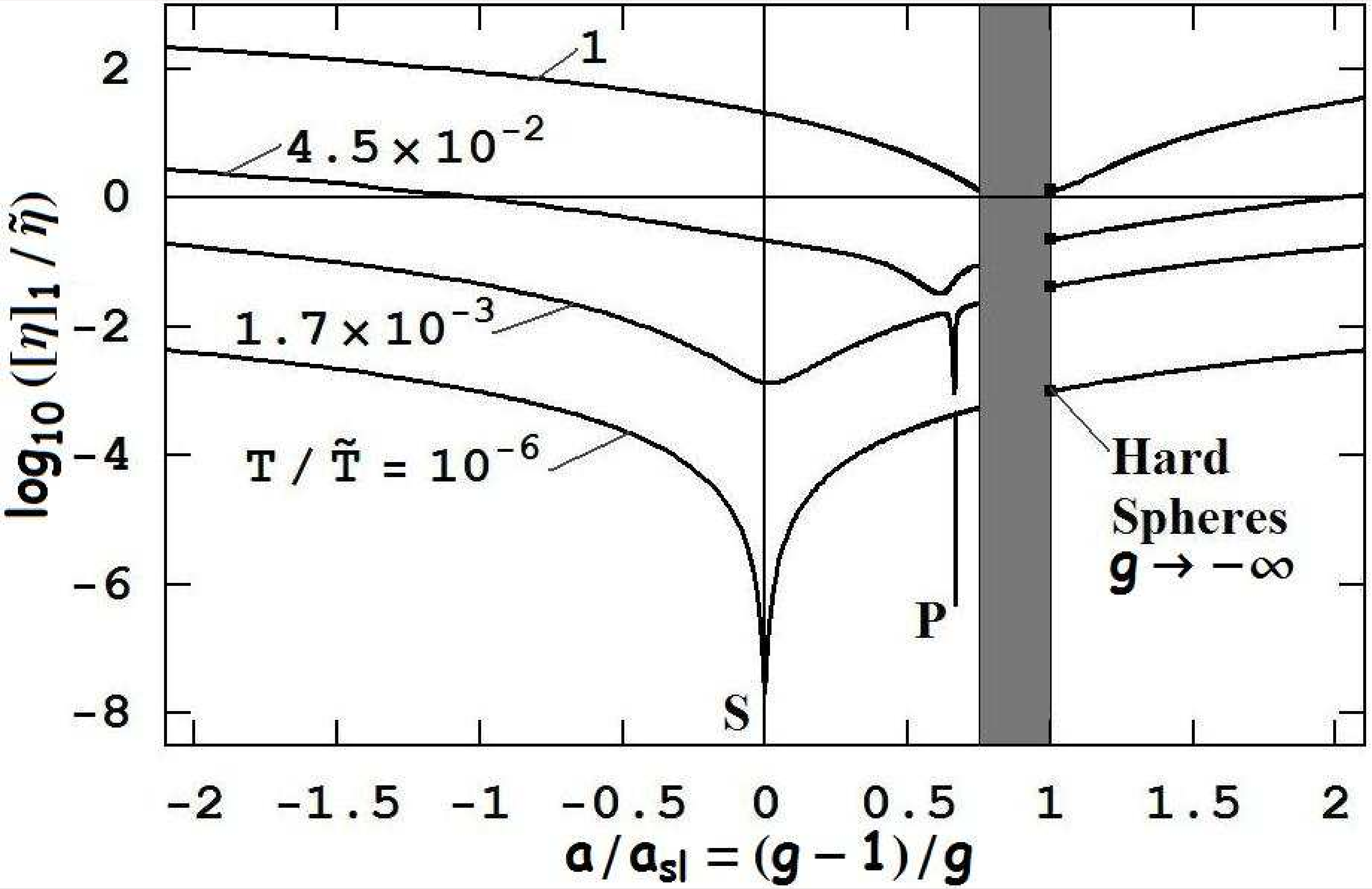
$$\frac{[\eta]_1}{\tilde{\eta}} = \left(\frac{a}{\lambda(T)}\right)\frac{1}{\omega^{(2,2)}(T)},$$

$$\frac{[\eta]_2}{[\eta]_1} = 1 + \frac{3(7\omega^{(2,2)}(T) - 2\omega^{(2,3)}(T))^2}{2\left(\omega^{(2,2)}(T)\,(77\omega^{(2,2)}(T) + 6\omega^{(2,4)}(T)) - 6\left(\omega^{(2,3)}(T)\right)^2\right)},$$

and symmetry correction: $\times\left(1 - n\lambda^3(T)\epsilon(T)\right)$

- For constant cross sections, the $\omega-$ integrals are $T-$independent; as a result $[\eta]_{1,2} \propto T^{1/2}$ as $\lambda(T) \propto T^{-1/2}$.

Viscosity vs inverse scattering length



Asymptotic trends of viscosity

$$\frac{\eta}{\tilde{\eta}} \rightarrow \begin{cases} \left(\frac{1-g}{g}\right)^2 \left(T/\tilde{T}\right)^{1/2} & \text{for } g \neq 1,3 \\ 6\pi \left(T/\tilde{T}\right)^{3/2} & \text{for } g = 1 \\ \frac{16}{111} \left(T/\tilde{T}\right)^{1/2} & \text{for } g = 3\,. \end{cases}$$

Characteristic temperature:

$$\tilde{T} \equiv \frac{2\pi\hbar^2}{k_B m a^2} \;\; \text{or} \;\; \frac{T}{\tilde{T}} = \left(\frac{a}{\lambda}\right)^2$$

Self-diffusion

$$\tilde{\mathscr{D}} = \frac{3h}{32\sqrt{2}\pi}\frac{1}{ma^3 n},$$

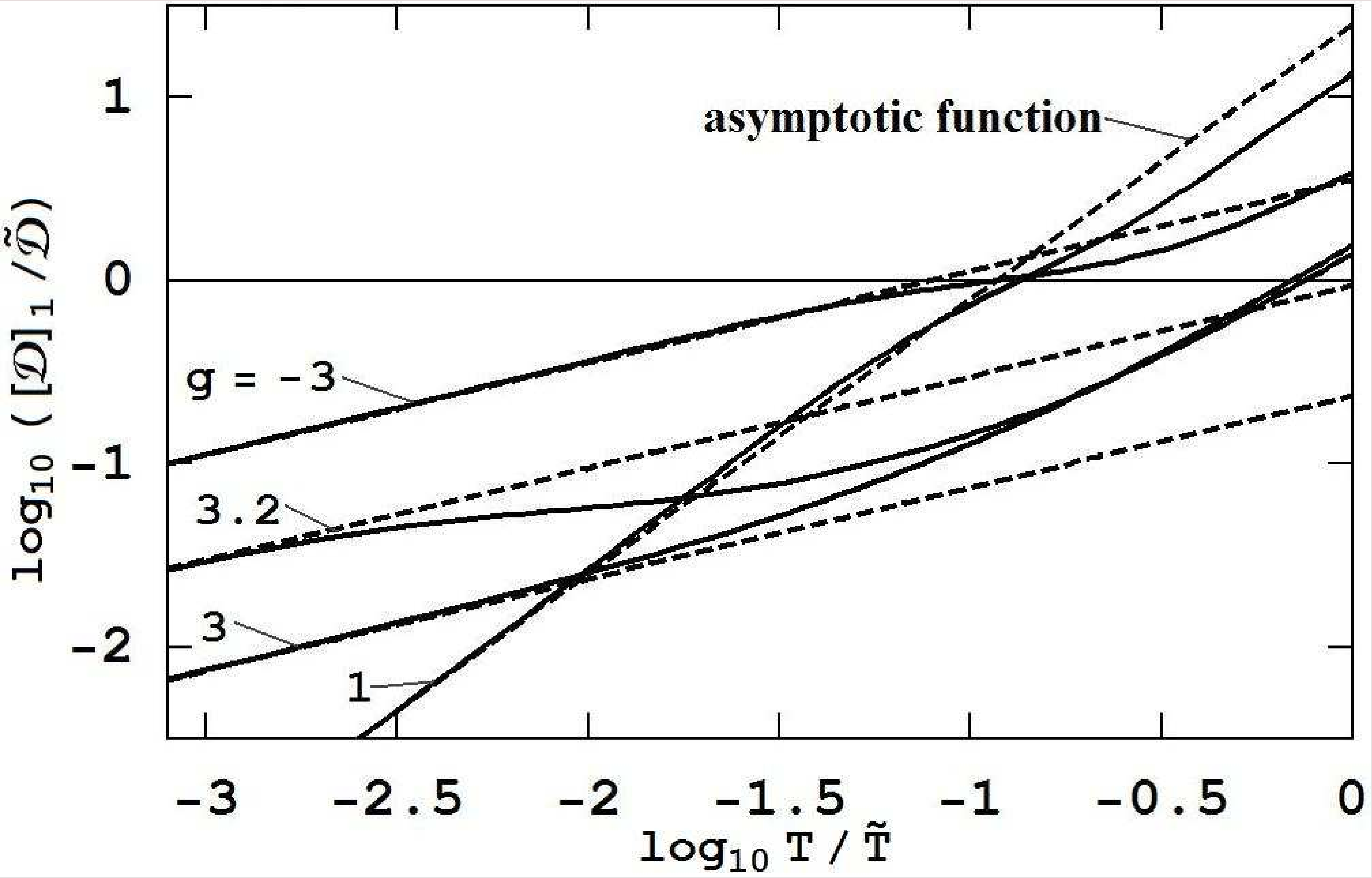
$$\frac{[\mathscr{D}]_1}{\tilde{\mathscr{D}}} = \left(\frac{a}{\lambda(T)}\right)\frac{1}{\omega^{(1,1)}(T)},$$

$$\frac{[\mathscr{D}]_2}{[\mathscr{D}]_1} = 1 + \frac{\left(5\omega^{(1,1)}(T) - 2\omega^{(1,2)}(T)\right)^2}{\omega^{(1,1)}(T)\left(30\omega^{(1,1)}(T) + 4\omega^{(1,3)}(T) + 8\omega^{(2,2)}(T)\right) - 4\left(\omega^{(1,2)}(T)\right)^2},$$

and symmetry correction: $\times\left(1 - n\lambda^3(T)\epsilon(T)\right)$

- For constant cross sections, the $\omega-$ integrals are $T-$independent; as a result $[\mathcal{D}]_{1,2} \propto T^{1/2}$ as $\lambda(T) \propto T^{-1/2}$.

Diffusion vs temperature



Asymptotic trends of diffusion

$$\frac{\mathcal{D}}{\tilde{\mathcal{D}}} \to 2\left(\frac{1-g}{g}\right)^2 \sqrt{\frac{T}{\tilde{T}}} \qquad \text{for} \qquad g \neq 1, 3\,.$$

$$\frac{\mathcal{D}}{\tilde{\mathcal{D}}} \quad \to \quad \begin{cases} 8\pi\left(T/\tilde{T}\right)^{3/2} & \text{for } g = 1 \\ \frac{6}{13}\left(T/\tilde{T}\right)^{1/2} & \text{for } g = 3\,. \end{cases}$$

Characteristic temperature:

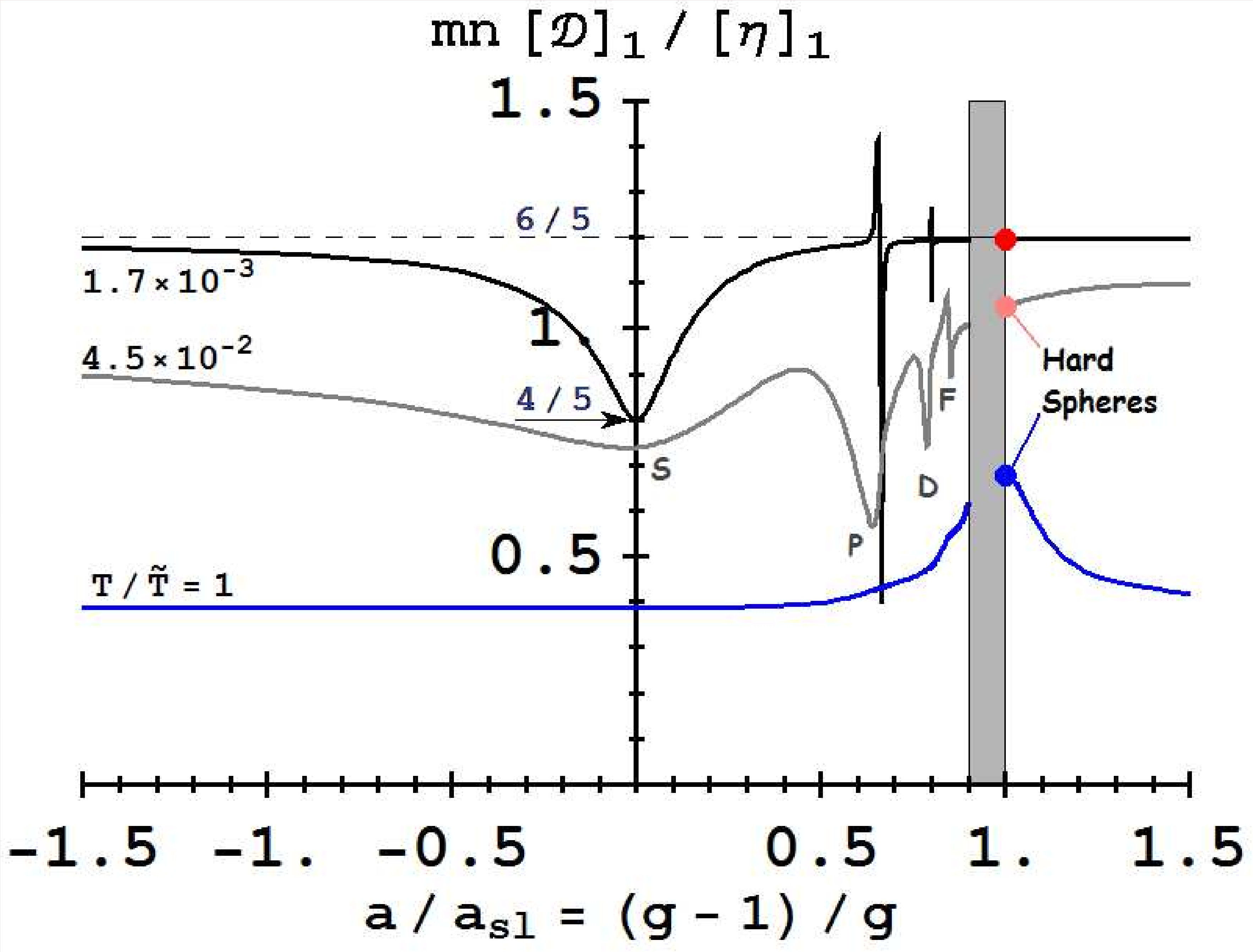
$$\tilde{T} \equiv \frac{2\pi\hbar^2}{k_B m a^2} \ \text{ or } \ \frac{T}{\tilde{T}} = \left(\frac{a}{\lambda}\right)^2$$

Effective physical volumes

g	$mn\mathcal{D}$	η	$mn\mathcal{D}/\eta$
1	$\frac{3\sqrt{2}\pi}{4}\frac{\hbar}{\lambda^3}$	$\frac{15\sqrt{2}\pi}{16}\frac{\hbar}{\lambda^3}$	$\frac{4}{5} = 0.80$
3	$\frac{9\sqrt{2}\pi}{104}\frac{\hbar}{\lambda\, a^2}$	$\frac{5\sqrt{2}\pi}{111}\frac{\hbar}{\lambda\, a^2}$	$\frac{999}{520} = 1.92$
$\neq 1, 3$	$\frac{3\sqrt{2}\pi}{8}\frac{\hbar}{\lambda\, a_{sl}^2}$	$\frac{5\sqrt{2}\pi}{16}\frac{\hbar}{\lambda\, a_{sl}^2}$	$\frac{6}{5} = 1.20$

Table 1: First order coefficients of diffusion (times mn), shear viscosity, and their ratios for $T \ll \tilde{T}$ for select g's .

Diffusion to viscosity ratio



Viscosity, η, to entropy density, s, ratio

- Is there a lower limit to η/s ?
- First proposal: $\eta/s \geq (4\pi)^{-1}\,(\hbar/k_B)$.
  Kovtun, Son & Starinets (2005)
- Recent works indicate even lower limits !
  Brigante et al. (2008), Buchel et al. (2008),
  Kats & Petrov (2009)
- What does the dilute delta-shell gas yield ?
- Is there anything deep in such a limit ?

Entropy density of a dilute delta-shell gas

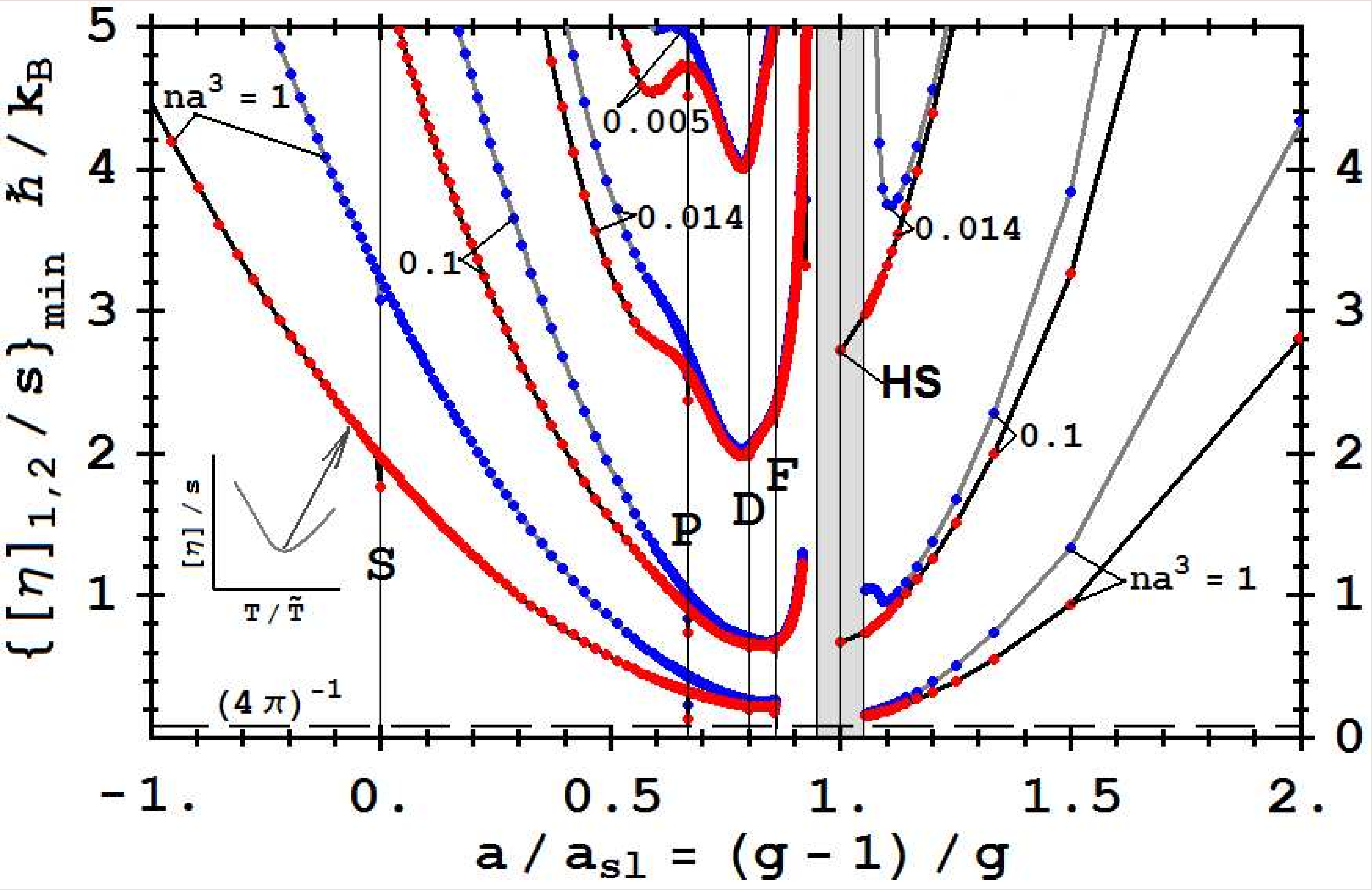
$$
\begin{aligned}
s &= (5/2 - \ln(n\lambda^3) + \delta s(T)\, na^3) n k_B \,, \\
\delta s(T) &= \left(\frac{a_2(T)}{2} - T \frac{da_2(T)}{dT} \right) \left(\frac{\lambda}{a} \right)^3 ,
\end{aligned}
$$

The second virial coefficient (that includes interactions)

$$
\begin{aligned}
a_2(T) = \mp\, 2^{-5/2} - 2^{3/2} {\sum_l}' (2l+1) \\
\times \left(e^{-E_l/(k_B T)} + \frac{1}{\pi} \int_0^\infty dx\, \frac{\partial \delta_l}{\partial x} e^{-\xi(T) x^2} \right) ,
\end{aligned}
$$

where the prime indicates summation over even l's for Bosons $(-)$ and odd l's for Fermions $(+)$, E_l is the energy of the bound state with angular momentum l and $\xi(T) = (\lambda/a)^2/(2\pi)$.

Viscosity to entropy density ratio



Lessons learned from the delta-shell gas

- Our analysis is restricted to the dilute gas limit, in which two particle interactions dominate but with scattering lengths that can take various values including infinity .
- Even at the two-body level, a rich structure in the temperature dependence and the effective physical volume responsible for the overall behavior of the transport coefficients are evident.
- The role of resonances in reducing the transport coefficients are amply delineated.
- Improved estimates of η and s have large roles on the ratio η/s !
- In the dilute gas limit, η/s for the delta-shell gas remains above $(4\pi)^{-1}\hbar/k_B$.
- Matching our results to those of intermediate and extreme degeneracies which highlight the additional roles of superfluidty and superconductivity reveals the extent to which many-body effects play a crucial role.


That's All Folks!